Constant approximation algorithms for embedding graph metrics into trees and outerplanar graphs

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Multiplicative and additive distortion

Distortion

- A metric space (X, d) is *embeddable with multiplicative distortion* $\lambda \geq 1$ into a host metric space (Y, d') if $\exists \varphi : X \mapsto Y$ such that $d(x, y) \leq d'(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y)$ for all $x, y \in X$.
- $\varphi : X \mapsto Y$ is an *embedding with additive distortion* $\lambda \geq 0$ if $d(x, y) \leq d'(\varphi(x), \varphi(y)) \leq d(x, y) + \lambda$ for all $x, y \in X$.
- Given a metric space (X, d) and a class $\mathcal{M}$ of host metric spaces, denote by $\lambda^* := \lambda^*(X, \mathcal{M})$ the *minimum (multiplicative or additive) distortion* of an embedding of (X, d) into a member of $\mathcal{M}$.

In this paper :

- Input metric spaces are the unweighted graphs $G = (V, E)$ endowed with shortest-path distance d_G;
- Host metric spaces are the class $\mathcal{T}$ of all *tree metrics* and the class $\mathcal{O}$ of all *outerplanar metrics* (metric spaces embeddable into a weighted tree or a weighted outerplanar graph).

Our main results

Tree metrics

Theorem 1 : There exists a factor 6 approximation algorithm with running time $O(|V||E|)$ for the optimal multiplicative distortion $\lambda^*(G, \mathcal{T})$ of embedding an unweighted graph G into a tree metric.

Outerplanar metrics

Theorem 2 : There exists a polynomial factor $100\lambda^* + 75$ approximation algorithm for the optimal multiplicative distortion $\lambda^* = \lambda^*(G, \mathcal{O})$ of embedding an unweighted graph G into an outerplanar metric.

Remark : This is the first result concerning the approximation of the optimal distortion of embedding into a class of metrics different from tree metrics.

Known results

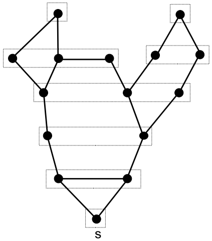
All results concern embeddings into trees

- Additive distortion : factor 6 approximation algorithm for general input metrics (3-approximation for l_∞-error norm) [Agarwala et al. 1999] (the result was simplified and related with Gromov's construction for δ-hyperbolic metric spaces in [Chepoi and Fichet, 2000]) ;
- Multiplicative distortion : factor exponential in $\sqrt{\log \Delta} / \log \log n$ (Δ is the aspect ratio) for embedding general metrics into tree metrics [Bădoiu, Indyk, and Sidiropoulos, 2007]
- Multiplicative distortion : factor 100 approximation algorithm for input graph metrics [Bădoiu, Indyk, and Sidiropoulos, 2007], improved to a factor 27 algorithm by [Bădoiu et al., 2008].

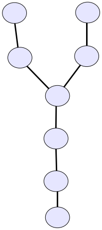
Main ingredients

- Tree metrics : Layering partition of G into clusters, lower bounding of $\lambda^*(G, \mathcal{T})$ by the maximal diameter D of a cluster, a construction of a (uniformly) weighted tree T and an embedding of G into T whose distortion is $O(D)$.
- Outerplanar metrics : Layering partition of G into clusters, partition of clusters into cells, lower bounding of $\lambda^*(G, \mathcal{O})$ using the concept of *metric relaxed minor* ($K_{2,3}$ in this case) and the maximal diameter D of a cell, and construction of an (weighted) outerplanar graph H into which G embeds with distortion $O(D)$.

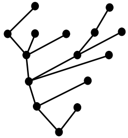
- A *layering* of $G = (V, E)$ with respect to a vertex s is the decomposition of V into the *spheres* $L^i = \{u \in V : d(s, u) = i\}$, $i = 0, 1, 2, \ldots$.
- A *layering partition* $\mathcal{L}P(s) = \{L^i_1, \ldots, L^i_{p_i} : i = 0, 1, 2, \ldots, r\}$ of G is a partition of each sphere L^i into *clusters* $L^i_1, \ldots, L^i_{p_i}$ such that $u, v \in L^i$ belong to L^i_j iff they can be connected by a path outside the ball $B_{i-1}(s)$.

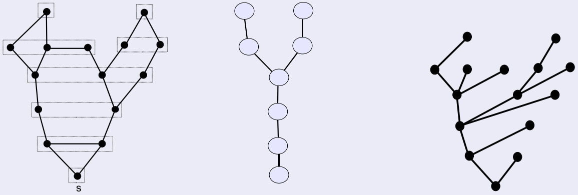


Layering tree Γ of G : the vertex set is the set of all clusters L^i_j in a layering partition $\mathcal{L}P$ of G. Two vertices $C = L^i_j$ and $C' = L^{i'}_{j'}$ are adjacent in Γ iff there exist $u \in L^i_j$ and $v \in L^{i'}_{j'}$ such that u and v are adjacent in G.



A *tree* $H = (V, F)$ (closely reproducing the global structure of Γ) : identify for each cluster $C = L^i_j \in \mathcal{L}P$ an arbitrary vertex $x_C \in L^{i-1}$ which has a neighbor in $C = L^i_j$ and make x_C adjacent in H with all vertices $v \in C$.

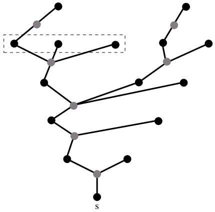




A layering partition $\mathcal{L}P$ of G and the trees Γ and H associated with $\mathcal{L}P$.

The trees H', H_w, and H'_w

- H' is obtained from H by introducing for each cluster C a Steiner point p_C, and adding edges of length 0 between p_C and vertices of C and edges of length 1 between p_C and x_C;
- H_w is obtained by assigning uniform weight $w := D + 1$ to edges of H;
- H'_w is obtained from H' by assigning uniform weight $w := (D + 1)/2$ to edges of H'.



The tree H'

Embedding into tree-metrics I

Properties of H_w, H' and H'_w

Lemma 1. For any two vertices of u, v of G,

- $d_G(u,v) \leq d_{H_w}(u,v) \leq (D+1)(d_G(u,v)+2)$;
- $d_{H'}(u,v) \leq d_G(u,v) \leq d_{H'}(u,v) + D$;
- $d_G(u,v) \leq d_{H'_w}(u,v) \leq (D+1)(d_G(u,v)+1)$.

Basic property of graphs λ-embeddable into tree metrics

Lemma 2. If a graph G λ-embeds into a tree, then the diameter of any cluster C of a layering partition $\mathcal{L}P$ is at most 3λ, i.e., $\lambda^*(G, \mathcal{T}) \geq D/3$.

Remark. Lemmas 1 and 2 already provide factor 12 (H_w) and factor 8 (H'_w) approximation algorithms (the trees H_w and H'_w are constructible in $O(|V||E|)$ time).

Embedding into tree-metrics II

To improve the factors 12 and 8 to 9 and 6, respectively, the following result is used :

Lemma 3. Let G λ-embeds into a tree, $C = L^i_j$ be a cluster of $\mathcal{L}P$ and $v \in C$. Then for any neighbor v' of v in previous level L^{i-1} and any $u \in C$, we have $d_G(v', u) \leq \max\{3\lambda - 1, 2\lambda + 1\}$.

In particular, if H_ℓ is the tree H weighted by $\ell := \max\{3\lambda - 1, 2\lambda + 1\}$ and H'_ℓ is the tree H' weighted by $\ell := 3\lambda/2$, then for any two vertices u, v of G,

- $d_G(u, v) \leq d_{H_\ell}(u, v) \leq \max\{3\lambda - 1, 2\lambda + 1\}(d_G(u, v) + 2)$;
- $d_G(u, v) \leq d_{H'_\ell}(u, v) \leq 3\lambda(d_G(u, v) + 1)$.

Embedding into tree-metrics III

Since we do not know λ in advance, we compute the weights of the trees H_ℓ and H'_ℓ in the same way as H_w and H'_w, but instead of weighting the edges with $w := D + 1$ and $w := (D + 1)/2$, the trees H and H' are weighted by the length function ℓ, where

- for H_ℓ, $\ell = \max\{d_G(u, v) : uv$ is an edge of $H\}$;
- for H'_ℓ, $\ell = \frac{1}{2}\max\{D, \max\{d_G(u, v) : uv$ is an edge of $H\}\}$.

Approximation by Tree Metric

Theorem 1. The tree-metrics of H_ℓ and H'_ℓ provide a 9 and a 6 approximation for the optimal distortion $\lambda^*(G, \mathcal{T})$.

Minors and relaxed minors I

Minors

A graph G is a *minor* of a graph G if H can be obtained from G by contracting some edges, deleting some edges, and deleting sone isolated vertices.

In particular, $H = (V', E')$ is a minor of $G = (V, E)$ if there exists a map $\mu : V' \cup E' \mapsto 2^V$, such that

(i) for any vertex v of H, $G(\mu(v))$ is connected ;

(ii) for any different vertices v, v' of H, $G(\mu(v)) \cap G(\mu(v')) = \emptyset$;

(iii) for any edge $e = uv$ of H, $G(\mu(e))$ is a path P_e of G with one end in $G(\mu(u))$ and another end in $G(\mu(v))$;

(iv) for any vertex v and any edge e of H with $v \notin e$, $P_e \cap G(\mu(v)) = \emptyset$;

(v′) for any two edges $e = (x, y), e' = (u, v)$ of H, the paths P_e and $P_{e'}$ intersect if and only if $\{x, y\} \cap \{u, v\} \neq \emptyset$ and if say, $e = (x, y), e' = (x, w)$ then P_e and $P_{e'}$ intersect only in $\mu(x)$.

Minors and relaxed minors II

Relaxed minors

$H = (V', E')$ is a *relaxed minor* of $G = (V, E)$ if there exists a map $\mu : V' \cup E' \mapsto 2^V$ satisfying (i)-(iv) and the following relaxation of (v′) :

(v) for any two non-incident edges e, e' of H, the paths P_e and $P_{e'}$ are disjoint.

The concept of relaxed minor is weaker than that of a minor : the 3-cycle is not a minor of any tree but is a relaxed minor of a 3-star.

Subdivided graphs

We conjecture that *if the graph H is triangle-free, then the notion of relaxed minor is not weaker than that of minor* and confirm it in case of *subdivided graphs*, i.e., graphs obtained from arbitrary graphs by subdividing their edges exactly once.

Proposition 1. If a graph $G = (V, E)$ has a subdivided graph $H = (V', E')$ as a relaxed minor, then G has H as a minor.

α-Metric relaxed minors I

α-Metric relaxed minors

$H = (V', E')$ is an *α-metric relaxed minor* of $G = (V, E)$ if there exists a map $\mu : V' \cup E' \mapsto 2^V$ satisfying (i)-(v) (i.e. H is a relaxed minor of G) and the following stronger version of (v) :

(v^+) for any two non-incident edges $e = uv$ and $e' = u'v'$ of H, the sets $\mu(u) \cup P_e \cup \mu(v)$ and $\mu(u') \cup P_{e'} \cup \mu(v')$ are α-far in G.

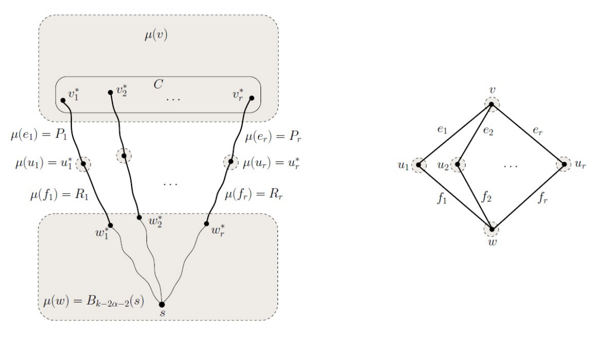
(Two sets A and B are α-far, $\min\{d_G(a, b) : a \in A, b \in B\} > \alpha$).

Proposition 2. If a subdivided 2-connected graph $H = (V', E')$ is an α-metric relaxed minor of a graph $G = (V, E)$, then any embedding of G into an H-minor free graph requires distortion $> \alpha$.

α-Metric relaxed minors II

Proposition 3. If for $\alpha > 1$ a cluster C of a layering partition of G contains $r \geq 3$ vertices $v_1^*, \ldots, v_r^*$ that are pairwise $(4\alpha + 2)$-far, then any embedding φ of G into a $K_{2,r}$-minor free graph has distortion $> \alpha$.

Proof of Proposition 3 :



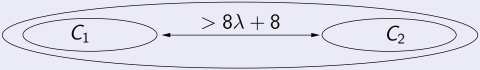
Outerplanar metrics I

Small, medium, and big clusters

- By Proposition 3, we will assume that $\lambda \geq 1$ is so that each cluster C has at most two $(4\lambda+2)$-far vertices.
- C is *bifocal* if it contains exactly two $(4\lambda+2)$-far vertices c_1 and c_2 ;
- The (Voronoi) *cells* of a bifocal cluster C are the sets
  $C_1 = \{x \in C : d_G(x, c_1) \leq d_G(x, c_2)\}$,
  $C_2 = \{x \in C : d_G(x, c_2) \leq d_G(x, c_1)\}$;
- C is a *small cluster* if $\mathrm{diam}(C) \leq 4\lambda + 2$;
- C is a *big cluster* if $\mathrm{diam}(C) > 16\lambda + 10$;
- C is a *medium cluster* if C is not small or big.

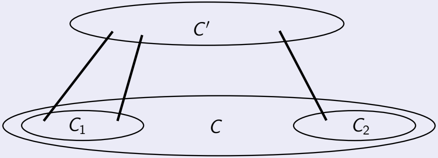
Outerplanar metrics II

Basic property of big clusters :



Spread clusters :

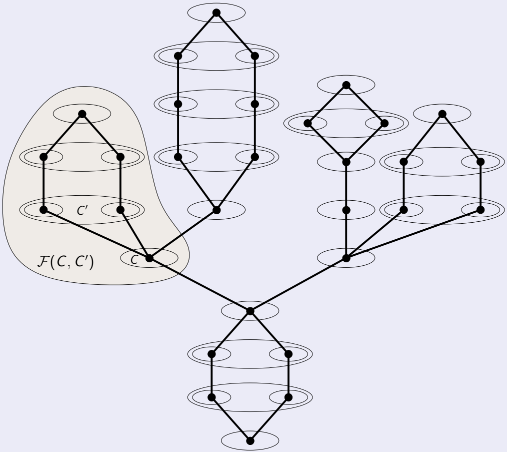
C' is *spread* if C' and its father C are bifocal and both cells C_1, C_2 of C are adjacent to C' :



Outerplanar metrics III

CC'-fiber :

The *CC'-fiber* $\mathcal{F}(C, C')$ consists of C and all clusters contained in the subtree of Γ rooted at C'



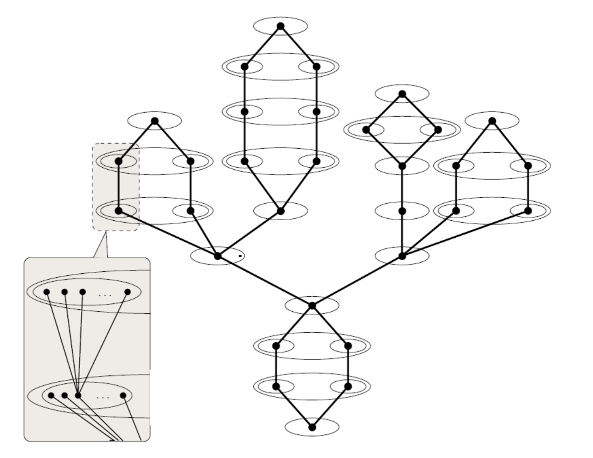
Outerplanar metrics IV

The sketch of the algorithm

- 1. If a cluster C has two big sons or C is big and has two spread sons, then the algorithm returns the answer "not" ;
- 2. So suppose that each cluster C has at most one spread son ;
- 3. Each small or medium cluster C is replaced by a star with center in its father ;
- 4. Each cell C_1 and C_2 of a big cluster C is also replaced by a star but the centers of these two stars are different vertices from the father (if the father is big, then these centers belong to different cells) ;
- 5. Medium clusters C are used to "start" new cycles (if C has a big son) and to "close" cycles (if C is the unique spread son of a big cluster) ;
- 6. Big clusters are used to "grow" cycles (if C is big and C has one big son C', then C, C' "contribute" with two edges to the cycle).

Outerplanar metrics V

An outerplanar graph produced by the algorithm



Outerplanar metrics VI

Approximation by Outerplanar Metric

1. **For** each cluster C of the layering partition **do**
2. **If** (a) C has two big sons or (b) C is big and has two spread sons, **return** the answer "not".
3. **Else** for each son C' of C **do**
4. *Case 1 :* **If** (a) C' is small, or (b) C' is medium and C is not big, or (c) C' is medium and not spread and C is big, **then** pick in C the center c of a cell of C adjacent to C' and make c adjacent to all vertices of C'.
5. *Case 2 :* **If** C' is medium, C is big, and C' is the (unique) spread son of C, **then** make the center c_1 of cell C_1 of C adjacent to all vertices of C'. Additionally, make the center c_2 of cell C_2 of C adjacent to every vertex of C'.
6. *Case 3 :* **If** C' is big with cells C'_1, C'_2, such that C'_1 is adjacent to C_1 and C'_2 is adjacent to C_2, where C_1 and C_2 are the cells of C with centers c_1 and c_2, **then** make c_1 adjacent to all vertices of C'_1 and c_2 adjacent to all vertices of C'_2.

Outerplanar metrics VII

Correctness I

Theorem 2. Let $G = (V, E)$ be an input graph and let $\lambda \geq 1$. If APPROXIMATION BY OUTERPLANAR METRIC returns "not", then any embedding of G into a $K_{2,3}$-minor free graph requires distortion $> \lambda$. Otherwise, if the algorithm returns the outerplanar graph $G' = (V, E')$, then assigning to its edges weight $w := 20\lambda + 15$, we obtain an embedding of G to G' such that $d_G(x, y) \leq d_{G'}(x, y) \leq 5wd_G(x, y)$ for any x, y of V. As a result, we obtain a factor $100\lambda + 75$ approximation of the optimal distortion of embedding a graph distance into an outerplanar metric.

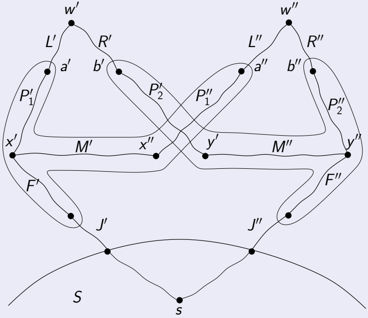
Proposition 4. For each edge xy of the graph G, the vertices x and y can be connected in the outerplanar graph G' by a path consisting of at most 5 edges, i.e. $d_{G'}(x, y) \leq 5w$. Conversely, for each edge xy of the graph G', we have $d_G(x, y) \leq 20\lambda + 15$.

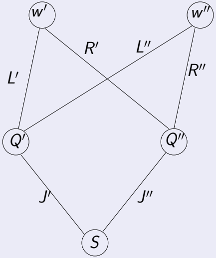
Outerplanar metrics VIII

Correctness II

Proposition 4. Let C be a big or an almost big cluster having two sons C', C'' such that the two cells of C can be connected in both CC'- and CC''-fibers of G. Then, any embedding of G in a $K_{2,3}$-minor free graph requires distortion $> \lambda$. These conditions are fulfilled in the two cases when the algorithm returns the answer "not".

To the proof of Proposition 4





Perspectives

Open questions

- Extend the approximation results from outerplanar metrics to $K_{2,r}$-free metrics and to K_4-free metrics (series-parallel metrics).
- (after [Bădoiu et al., 2008]) Is it possible to approximate the multiplicative distortion of embedding of an arbitrary metric in a tree-metric by a polylogarithmic factor ?
- Prove that if the graph H is triangle-free, then the notion of relaxed minor is not weaker than that of minor.