Topological Superrigidity

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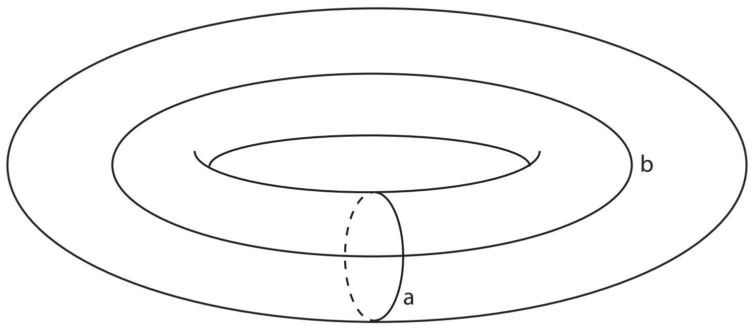
An elementary observation

Every π_1-injective map $f : S^1 \to S^1 \times S^1$ factorises up to homotopy as a finite cover of an embedding.

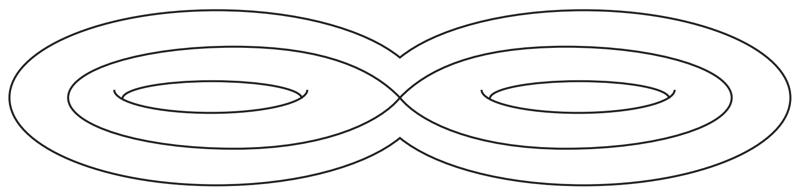


A curve representing the element $a^r b^s$ is, up to homotopy a finite cover of the embedded curve representing $a^{r/d} b^{s/d}$ where $d = \gcd(r, s)$

Hyperbolic surfaces

On the other hand every hyperbolic surface Σ admits π_1-injective maps $f : S^1 \to \Sigma$ which do not factorise up to homotopy as a finite cover of an embedding.

- Geodesics minimise self intersection number in their free homotopy class.
- Intersection numbers are additive with exponents.



Choose a closed geodesic with non-trivial self-intersection number and intersection number 1 with some scc.

Compare the situation with surfaces in 3-manifolds

Theorem (The Kahn-Markovic theorem)

Every hyperbolic 3-manifold contains an immersed π_1-injective surface.

Theorem (Alan Reid)

There is a hyperbolic 3-manifold which is finitely covered by a surface bundle over S^1 but which does not contain any embedded π_1-injective surfaces.

The low genus embedding theorems

Theorem (Waldhausen's torus theorem)

Let N be the 2-torus, M be a closed, aspherical, orientable 3-manifold which is not Seifert fibered and let f be a π_1-injective map from N to M. Then there is a π_1-injective embedding of N in M.

Theorem (The sphere theorem)

Let N be the 2-sphere, M be a closed, orientable 3-manifold and let f be a π_2-injective map from N to M. Then there is a π_2-injective embedding of N in M.

Whitney's embedding theorem

The embedding obstruction vanishes if the codimension is high enough:

Theorem (Whitney)

Let N be an n-manifold and M be an m-manifold with either $2n+1 \leq m$ *or* $m = 2n > 6$ *and* $\pi_1(M) = 1$. *Then any map f from N to M is homotopic to an embedding.*

Geometric Superrigidity

Theorem (Ngaiming Mok, Yum-Tong Siu, Sai-Kee Yeung, Inventiones 1993)

Let $\tilde{N}$ be a globally symmetric irreducible Riemann manifold of non-compact type.

Let $\tilde{M}$ be a Riemann manifold. Let f be a non-constant H-equivariant harmonic map from $\tilde{N}$ to $\tilde{M}$.

f is a totally geodesic isometric embedding (up to a renormalization constant).

Theorem (The torus theorem)

π_1-injective, codimension-1 $\longrightarrow$ embedding, up to cut and paste

Theorem (Whitney's embedding theorem)

Continuous $\longrightarrow$ embedding, up to homotopy.

Theorem (Geometric Superrigidity)

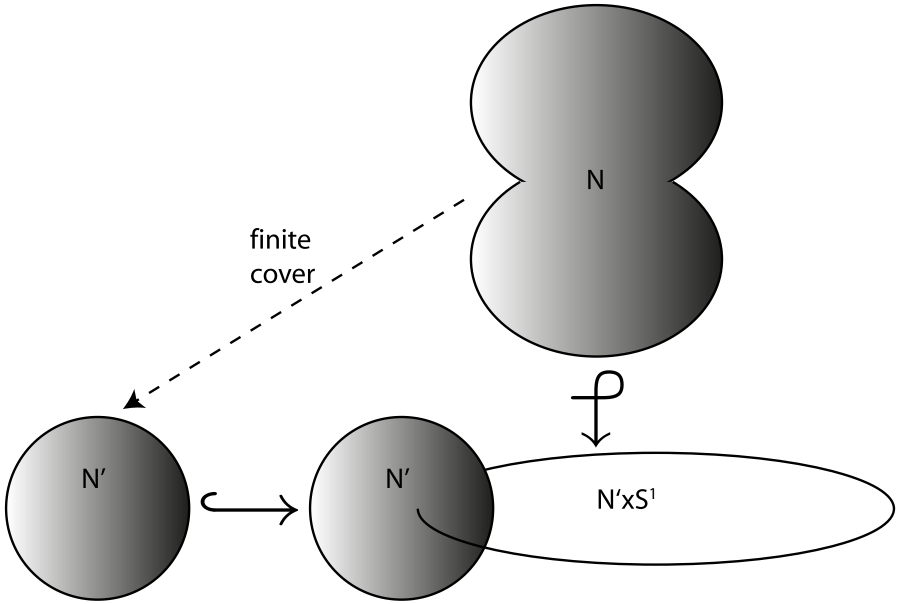
Harmonic $\longrightarrow$ totally geodesic embedding, up to renormalization.

Theorem (Kar, GAN + GAN, Reeves)

Let N be an orientable n-manifold with universal cover $\tilde{N}$ a globally symmetric irreducible Riemann manifold of non-compact type.
Assume that either $\tilde{N}$ is of rank at least 2, or $\tilde{N}$ is the quaternionic or Cayley hyperbolic space.
Let M be a closed, orientable, aspherical $n+1$ dimensional manifold and let f be a π_1-injective map from N to M. Then f is homotopic to a finite cover of an embedding $N' \hookrightarrow M$ and N' admits a metric modelled on the symmetric space $\tilde{N}$.

$$\begin{array}{ccc} & & N \\ & \overset{p}{\swarrow} & \downarrow f \\ N' & \hookrightarrow & M \end{array}$$

Example



If M' is an $n+1$ manifold which has boundary consisting of two π_1-injective copies of N' then any diffeomorphism between them gives rise to a closed manifold and p induces an immersion $N \to M$ as required.

In many cases, e.g., when N is quaternionic or Cayley hyperbolic, these are the **only** examples.

Outline of the proof

Step 1 (Geometric group theory) Replace $H < G$ with a subgroup $H' < G$ commensurable to H and such that G splits over H, $G = A *_{H'} B$ or $G = A*_{H'}$ with $H < A$.

Step 2 (Surgery theory) Apply Cappell's surgery lemmas to realise the splitting by an embedded submanifold $i : N' \hookrightarrow M$ so that i_* is π_k-injective for all $k \leq n/2$.

Step 3 (Homological algebra) Appeal to Poincaré duality to conclude that i_* is π_k-injective for all k and that $H < H'$.

Step 4 (Algebraic Topology/Rigidity) Conclude that the map f factors up to homotopy through a finite cover of the embedding i.

Step 1: Generalising Stallings' theorem

Theorem (GAN)

Let G be a finitely generated group and $H < G$ satisfy

- $e(G, H) \geq 2$,
- $Sing(G, H) \subset Comm_G(H)$.

Then G splits over a subgroup commensurable with H

$$G = A \underset{H'}{*} B \text{ or } A \underset{H'}{*} .$$

The proof is a version of the Casson/Dunwoody least weight track argument carried out on a cube complex. It is easy to see that we can arrange that $H < A$.

Theorem (Kar, GAN)

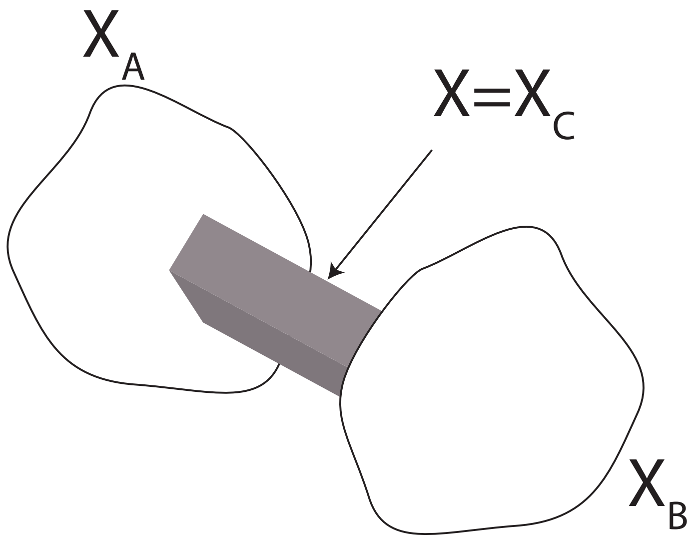
Let G be an orientable Poincaré duality group of dimension n. Suppose that H is an orientable $(n-1)$-dimensional Poincaré duality subgroup of G and that H has property (T). Then G splits over a subgroup commensurable with H.

- By Poincaré duality $e(G, H) \geq 2$.
- G acts on a CAT(0) cube complex with H as the hyperplane stabiliser.
- Since H has property (T) it must, by (Reeves, GAN) fix a point in the hyperplane and so by an argument of Kropholler if $g \in \mathrm{Sing}(G, H)$ then $H \cap H^g$ is also a codimension-1 subgroup of G.
- $cd(H \cap H^g) = n - 1 = cd(H)$.
- By Strebel's theorem $[H : H \cap H^g] < \infty$, so g lies in the commensurator of H.

We can then apply the generalised Stallings' Theorem to obtain the required splitting.

The Scott-Wall $K(\pi, 1)$

The subgroups A, C (and B) have geometric dimension n so we choose a cell complex of dimension n to realise each of them, and build the Scott-Wall $K(\pi, 1)$ for G:



The Scott-Wall complex Y

Step 2a, Cappell's surgery lemmas part 1

Lemma

Let Y be an $(n+1)$-dimensional closed manifold (or Poincaré complex) and X a codimension-1 closed submanifold (or sub-Poincaré complex) with trivial normal bundle in Y and with $\pi_1(X) \to \pi_1(Y)$ injective. Let M be an $(n+1)$-dimensional closed differentiable (or PL) manifold with $f : M \to Y$ a homotopy equivalence, $n \geq 4$[a]*. Then f is homotopic to a map, which we continue to call f, which is transverse regular to X (whence $f^{-1}(X)$ is a codimension-1 submanifold of M) and with the restriction of f to $f^{-1}(X) \to X$ inducing isomorphisms $\pi_i(f^{-1}(X)) \to \pi_i(X)$, $i \leq (n-1)/2$.*

[a] Without the assumption that M is differentiable or PL Cappell's result applies only for $n \geq 5$.

Step 2b, Cappell's surgery lemmas part 2

We give a simplified statement:

Lemma

Assume further that $f : M \to Y$ *is a homotopy equivalence transverse regular to* X *with* $\pi_i(f^{-1}(X)) \to \pi_i(X)$ *an isomorphism for* $i < n/2$*. Then* $\exists [P] \in \tilde{K}_0(H)$ *such that if* $[P] = 0$ *then* f *is homotopic to a map* f' *with* $f'^{-1}(X)$ *n/2-connected.*

We can apply this lemma in our context since vanishing of the **entire** reduced projective class group $\tilde{K}_0(H)$ is provided by Bartels and Lück in their proof of the Borel Conjecture for hyperbolic and CAT(0) spaces.

Step 3, Poincaré duality

Lemma

Let N' be a closed orientable $2k$ dimensional manifold such that its universal cover $\tilde{N}'$ is k-connected. Suppose moreover that $G = \pi_1(N')$ is a $2k$-dimensional Poincaré duality group. Then $\pi_i(N') = \{0\}$ for all $i \geq 2$.

Proof.

Apply the Hurewicz isomorphism to the smallest degree for which $\pi_n(\tilde{N}') \neq \{0\}$.

$$\{0\} \neq \pi_n(\tilde{N}') \overset{Hurewicz}{=} H_n(\tilde{N}') \overset{duality}{=} H_c^{2k-n}(\tilde{N}') = H^{2k-n}(G, \mathbb{Z}G) \overset{duality}{=} \{0\}.$$

□

Now let $N' = f'^{-1}(X)$. Combining the results above we see that N' is an aspherical n-dimensional submanifold $N' \hookrightarrow M$ which induces the splitting of $\pi_1(M)$ over $H' = \pi_1(N')$.

Step 4a, Mostow-Prasad-Farb rigidity

- $\pi_1(N')$ is commensurable with the lattice $\pi_1(N)$ in an isometry group satisfying Mostow-Prasad-Farb rigidity.
- It follows that, modulo a finite subgroup, $\pi_1(N')$ is a uniform lattice in the same isometry group.
- On the other hand $\pi_1(N')$ is torsion free, so it is in fact a lattice and by rigidity N' admits a metric modelled on $\tilde{N}$.

Step 4b, homological algebra

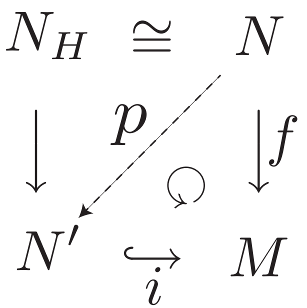
Recall that $\pi_1(N')$ and $\pi_1(N)$ are commensurable subgroups of A. In most cases $\pi_1(N')$ is self commensurating so that $\pi_1(N) < \pi_1(N')$ as required.

Lemma

Let (A, Ω) be a Poincaré duality pair, with H' the stabiliser of a point in Ω. Then either H' is self commensurating in A, or it is of twisted I-bundle type, i.e., the index $[A : H'] = 2$ and A acts non-trivially on Ω.

If (A, Ω) is of twisted I-bundle type then A is a non-orientable Poincaré duality group of dimension n and contains a unique maximal orientable PD^n subgroup. Since H' has index 2 and is an orientable PD^n subgroup it is the maximal one, and since H is also an orientable PD^n subgroup it is contained in H' as required.

Since $H < H'$, there is a covering space $N_H \longrightarrow N'$ induced by the inclusion.



Applying Borel rigidity we see that $N_H \cong N$.
This gives a finite covering map $N \longrightarrow N'$ and the composition $p \circ i : N' \longrightarrow M$ induces the inclusion of $\pi_1(N)$ in $\pi_1(M)$.

Since M is aspherical the map $p \circ i$ is homotopic to the map f as required. $\square$

A corollary

Recall the following classical fact:

Theorem

If M^{4d+1} is a differentiable manifold such that the first Betti number $b_1(M)$ of M is 0 and N^{4d} has non-zero signature then there are no codimension-1 immersions of N into M.

Proof.

Since f is a codimension-1 immersion, $f^* : H^{4d}(M, \mathbb{Q}) \to H^{4d}(N, \mathbb{Q})$ maps the Hirzebruch L-class $L_d(M)$ onto $L_d(N)$. It follows from Poincaré duality that

$$H^{4d}(M, \mathbb{Q}) \cong H_1(M, \mathbb{Q}) = \{0\}.$$

However Hirzebruch's signature theorem says that $L_d(N)$ is equal to the signature of N which is non-trivial. □

Using non-vanishing of Pontryagin numbers instead we get the following generalisation:

Theorem (Kar, GAN)

WIth the assumptions of the topological superrigidity theorem, if N has a non-trivial Pontryagin number[a] and M has first Betti number 0 then there are no π_1-injective maps $f : N \to M$.

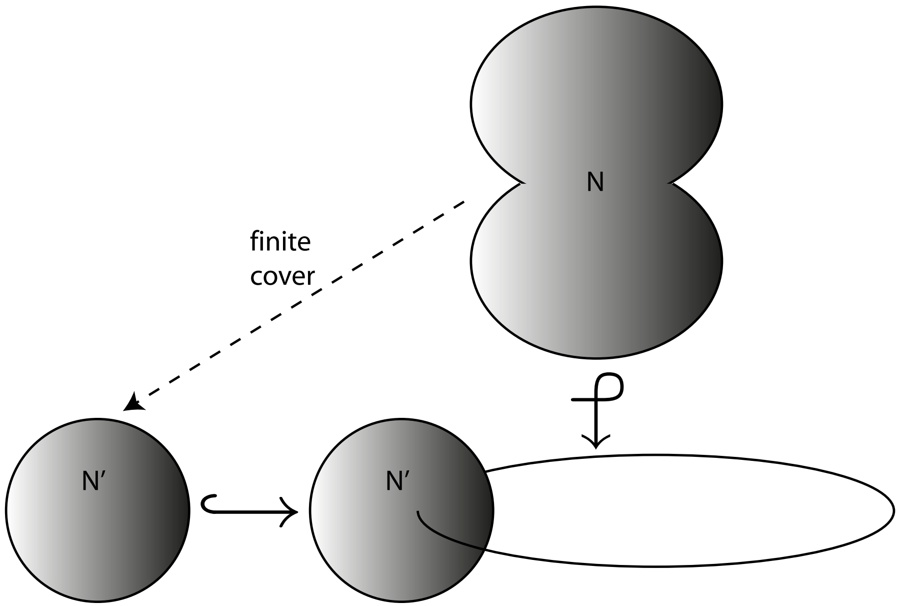
[a] E.g., by Lafont and Roy: orientable Cayley hyperbolic manifolds or orientable quaternionic manifolds of dimension at least 8.

- This works when *f* is not an immersion and when the dimension of *N* is not divisible by 4, both of which are prerequisites for the Hirzebruch theorem.
- If *M* is a suitable non-positively curved manifold then the geometric superrigidity theorem theorem tells us that since there are no π_1-injective maps, there are **no** non-trivial maps $f : N \to M$.

Proof.

By the topological superrigidity theorem we can replace $f : N \to M$ by an embedding $i : N' \hookrightarrow M$ finitely covered by f, and since $\beta_1(M) = 0$ the image is separating. But then $i(N')$ bounds orientably, so N' is null-cobordant.
On the other hand N has a non-trivial characteristic class (at least one of its Pontryagin numbers is non-trivial) and these vary multiplicatively with degree of finite coverings
It follows that N' also has a non-trivial Pontryagin number. This is a contradiction. □

Earlier example revisited



It follows that when N is quaternionic or Cayley hyperbolic $\pi_1(M)$ splits as an HNN extension and M is obtained from a manifold with two copies of N in the boundary, identified by a homeomorphism.