Artin groups: automatic structures and geodesics

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Oxford, 12th November 2010

Aims of the talk

I report on my result with Holt that any Artin group of large type in its natural presentation

- is shortlex automatic, given any ordering of the generators, and
- its set of geodesics satisfies the falsification by fellow traveller property (FFTP), and so is regular.

I'll start with some general info about automatic groups, and how to prove automaticity.

I'll demonstrate a mechanism that rewrites words in an Artin group of large type to shortlex minimal form, and explain why we need largeness.

I explain how we prove that this gives an automatic structure, then how we verify the FFTP.

Notation

$G = \langle X \mid R \rangle$ is a group with finite generating set X, finite set R of relators.

$\Gamma(G, X)$ is the Cayley graph for G over X.

$X^{\pm} = X \cup X^{-1}$.

A word over X is a string over $X^{\pm}$, and $(X^{\pm})^*$ is the set of all words.

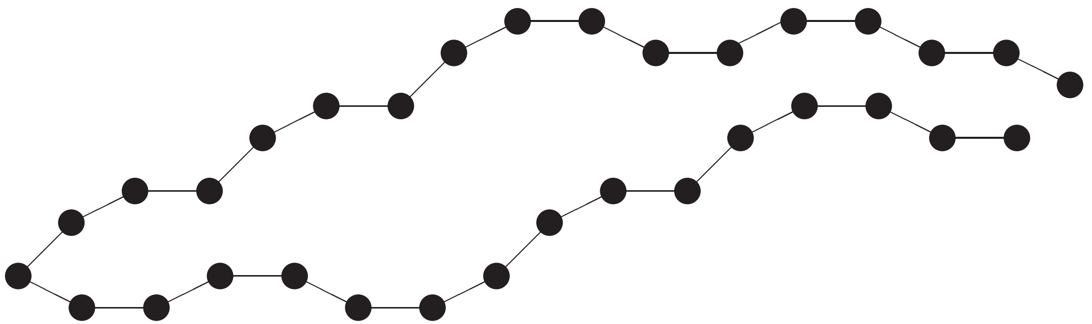
$=$ is used for equality of strings, $=_G$ for equality of group elements, $|.|$ for word length.

The shortlex word order puts u before v ($u <_{\text{slex}} v$) if either $|u| < |v|$ or $|u| = |v|$ but u precedes v lexicographically.

Automatic groups: an introduction

G is **automatic** if there is a regular set L of words over X, mapping onto G, and $k \in \mathbb{N}$ s.t. any $v, w \in L$ with $v =_G w$ or $v =_G wx$, $x \in X^{\pm}$, k-fellow travel.

L is regular if '$w \in L$?' can be decided by reading w sequentially using bounded memory, i.e. with a finite state automaton, FSA.



Two words v, w **k-fellow travel** ($v \sim_k w$) if the paths they trace out from the origin of $\Gamma(G, X)$ stay at most k-apart through their length.

We call L the **language** of the automatic structure, k the **fellow travelling constant** of L.

The definition of an automatic group was given by Thurston, built on work of Jim Cannon, who identified properties of the fundamental groups of compact hyperbolic manifolds that Thurston phrased in terms of fsa.

An automatic structure for G gives us a tool to compute within G.

- Thurston et al. were interested in plotting images of actions of their automatic groups.
- Proof in 1995 by Holt (using Warwick software) that Fibonacci group $F(2,9)$ was automatic led immediately to an alternative proof of its infiniteness.

An automatic group G is **shortlex automatic** if L can be chosen to be the minimal reps of group elements under the shortlex word order.

Examples of automatic groups

- $\pi_1(M)$, for M compact and hyperbolic (Jim Cannon).
- Free groups in standard presentation , abelian groups, virtually abelian groups, but no other nilpotent groups (Holt)
- $\pi_1(M)$, M any compact 3-manifold except those based on Nil or Sol, or any finite volume hyperbolic manifold (Epstein et al).
- Coxeter groups (Brink-Howlett)
- braid groups (Thurston)
- Artin groups of spherical type (Charney), 3-generated of large type (McCammond and Brady), extra large type (Peifer)

Definition is independent of generating set, class is closed under finite extensions, direct products, free products, and more.

Different ways to prove a group G automatic

Through the combinatorics of a presentation.

- If G is word hyperbolic, then there's an automatic structure on the language of geodesic words (any gen. set).
- Certain small cancellation properties imply G automatic.

Proper co-compact action of G on a fin. dim. CAT(0) cube complex $\Rightarrow$ biautomatic (automatic with additional fellow traveller property).

Through construction of a candidate language, then verification that it is part of an automatic structure.

- using clever ideas, e.g. braid groups (Thurston), Garside groups (Charney, Dehornoy).
- using Warwick software (for shortlex automatic) and its variants.

Proving a group SL automatic using the Warwick software

Step 1: We compute a (probably incomplete) rewrite system $\mathcal{R}$ for G, reducing wrt shortlex, then a finite set $\mathcal{D}$ of **word differences**, which measure fellow travelling between words u, v of rules $u \to v$.

Words u, v $\mathcal{D}$-fellow travel if all word differences between them are in $\mathcal{D}$; we write $u \sim_{\mathcal{D}} v$. $\delta(u, v)$ is the final difference, representing $u^{-1}v$.

Step 2: An FSA is constructed to recognise the set

$$L = \{w : \nexists v,\ v <_{\mathrm{slex}} w,\ v \sim_{\mathcal{D}} w,\ \delta(v, w) = \epsilon\}$$

Step 3: We check if L *is* the language of an automatic structure.

L is certainly regular, prefix closed, contains the shortlex minimal representative of any group element. Standard computations on automata reveal whether it **only** contains shortlex minimal words, and satisfies the fellow traveller property.

Axiom checking (step 3)

We define regular sets L_g for $g = e$ or x^ϵ $(x \in X, \epsilon = \pm 1)$ by

$L_e := \{(w, w) : w \in L\}, \quad L_{x^\epsilon} := \{(u, v) : u, v \in L,\ u \sim_D v,\ \delta(u, v) = x^\epsilon\}.$

and see that the definition of L ensures that

$$L_{x^{-1}} = \{(u, v) : (v, u) \in L_x\}$$

Now for subsets A, B of $L \times L$, we define

$$A \circ B = \{(a, b) : \exists c, (a, c) \in A, (c, b) \in B\}.$$

We see that $\circ$ is associative on $\{L_g : g \in \{e\} \cup X^{\pm 1}\}$ and so we can define L_w unambiguously for $w \in (X^{\pm})^*$ by

$$L_{w_1 w_2} = L_{w_1} \circ L_{w_2}.$$

We observe that

$$L_e \circ L_w = L_w \circ L_e = L_w.$$

All the sets L_w are regular, and the automata to recognise them can be constructed.

Using standard computations on automata we can check if
(a) for each $g \in X^{\pm}$, $L_{gg^{-1}} = L_e$, and
(b) for each defining relator r for G, $L_r = L_e$.

Provided that (a) and (b) hold, G is proved shortlex automatic.

Why axiom checking works

$$\begin{aligned}(a),(b) \Rightarrow\ & L_{grg^{-1}} = L_{gr} \circ L_{g^{-1}} = (L_g \circ L_r) \circ L_{g^{-1}} = (L_g \circ L_e) \circ L_{g^{-1}} \\ =\ & L_g \circ L_{g^{-1}} = L_e, \\ \text{Hence} \quad & u =_G e \Rightarrow L_u = L_{u_1 r_1 u_1^{-1} \cdots u_k r_k u_k^{-1}} = L_e.\end{aligned}$$

If $w =_G v$, with $v = x_1 \cdots x_r, w = y_1 \cdots y_s \in L$, then $w =_G vv^{-1}w$, and where $v_i := x_1 \cdots x_i$, and $w_j = y_1 \cdots y_j$, we have

$$v_i \in L, w_j \in L, \quad v_i \sim_D v_{i-1}, \quad w_{j-1} \sim_D w_j.$$

So $(v, w) \in L_{v^{-1}w} = L_e = \{(u, u) : u \in L\}$. Hence $v = w$. So L contains a unique, shortlex minimal representative of each group element.

From $L_{gg^{-1}} = L_e$ we deduce that any element w of L fellow travels with some representative of wg. Since representatives are unique this implies the fellow traveller property, with k the maximal length of a word in D.

Rewriting to shortlex normal form using L and $\mathcal{D}$

We can use $\mathcal{D}$ to construct a function ρ that maps any word to the element of L that represents it.

We define $\rho(w) = w$ if $w \in L$.

For $w \in L$ but $wg \notin L$, we define $\rho(wg)$ to be the minimal representative under shortlex of the set

$$\{v : wg \sim_{\mathcal{D}} v, \delta(wg, v) = e\},$$

that is the unique element in the intersection of that set with L. For other words w we define $\rho(w)$ by iteration, so that

$$\rho(x_1 \cdots x_r) = \rho(\rho(x_1 \cdots x_{r-1})x_r)$$

We see that

$$\rho(v) = w \iff (e, w) \in L_v$$

Artin groups

An Artin group is defined in terms of a Coxeter matrix, i.e. a symmetric matrix $(\mathsf{m_{ij}})$ with entries in $\mathbb{N} \cup \{\infty\}$, and off-diagonal entries all at least 2. It has a presentation

$$\langle a_1, \ldots, a_n \mid {}_{\mathsf{m_{ij}}}(a_i, a_j) = {}_{\mathsf{m_{ij}}}(a_j, a_i) \quad \text{for each} \quad i \neq j \rangle,$$

We write ${}_m(a, b)$ for the alternating product $aba \cdots$ of length m, and $(a, b)_m$ for $\cdots bab$.

Adding the relations $a_i^2 = 1$ defines the associated Coxeter group.

An Artin group is said to be of spherical or finite type if the Coxeter group is finite, and of **dihedral** type if it is 2-generated.

It is of **large** or extra-large type if all $\mathsf{m_{ij}}$ are at least 3, or at least 4.

What was known...

- Braid groups are biautomatic; direct construction (Thurston).
- Artin groups of spherical type are biautomatic; generalisation of Thurston's construction (Charney). All Garside groups are biautomatic (Dehornoy).
- Artin groups of extra large type are biautomatic, direct construction (Peifer).
- Artin groups of large type are biautomatic; appropriate small cancellation (Brady, McCammond).
- Right angled Artin groups are biautomatic; appropriate action on CAT(0) cube complex (Hermiller and Meier).
- For dihedral Artin groups over Artin generators both set of all geodesics and a set of unique geodesic reps are regular. (Mairesse, Mathéus)

...and what's new

Theorem (Holt, Rees, 2010)

If G is an Artin group of large type then G is shortlex automatic with respect to its standard generating set, with any ordering on its generators.

What's interesting about our result is that we have a shortlex structure over the standard generating set. The automatic groups software would give this result, group by group. But our proof is combinatorial, and gives a clear rewrite function to shortlex normal form.

Dihedral Artin groups

The geodesics of all dihedral Artin groups have been studied by Mathéus and Mairesse. Let DA_m be the dihedral Artin group

$$DA_m = \langle a, b \mid {}_m(a, b) = {}_m(b, a) \rangle$$

For any word w over a, b, we define $p(w)$, $n(w)$ as follows:

$p(w)$ is the minimum of m and the length of the longest positive alternating subword in w,

$n(w)$ is the minimum of m and the length of the longest negative alternating subword in w.

Example:
For $w = aba^{-1}baba^{-1}bababa^{-1}b^{-1}$, in DA_3, we have $p(w) = 3$, $n(w) = 2$. The following theorem tells us that w is non-geodesic.

Theorem (Mairesse,Mathéus, 2006)

In a dihedral Artin group DA_m, a word w is geodesic iff $p(w)+n(w) \leq m$, and is the unique geodesic representative of the element it represents if $p(w) + n(w) < m$.

The **Garside element** Δ in a dihedral Artin group is used in rewriting. Δ is represented by ${}_m(a, b)$ and either Δ or Δ^2 is central (depending on whether m is even or odd. If m is odd, Δ conjugates a to b.

We write δ for the permutation of $\{a, b, a^{-1}, b^{-1}\}^*$ induced by conjugation by Δ.

Examples:

In DA_3, $\Delta = aba =_G bab$. $a^\Delta = b$, and $\delta(ab^3a^{-1}) = ba^3b^{-1}$

In DA_4, $\Delta = abab =_G baba$. $a^\Delta = a$, and $\delta(ab^3a^{-1}) = ab^3a^{-1}$

Critical words in dihedral Artin groups.

A freely reduced word w with $p(w) = p$, $n(w) = n$, $p+n = m$, is critical if it has the form

$$_p(x, y)\xi(z^{-1}, t^{-1})_n \quad \text{or} \quad _n(x^{-1}, y^{-1})\xi(z, t)_p,$$

where $\{x, y\} = \{z, t\} = \{a, b\}$. (Have to be a bit careful if p or n is zero.) We define an involution τ on the set of critical words by

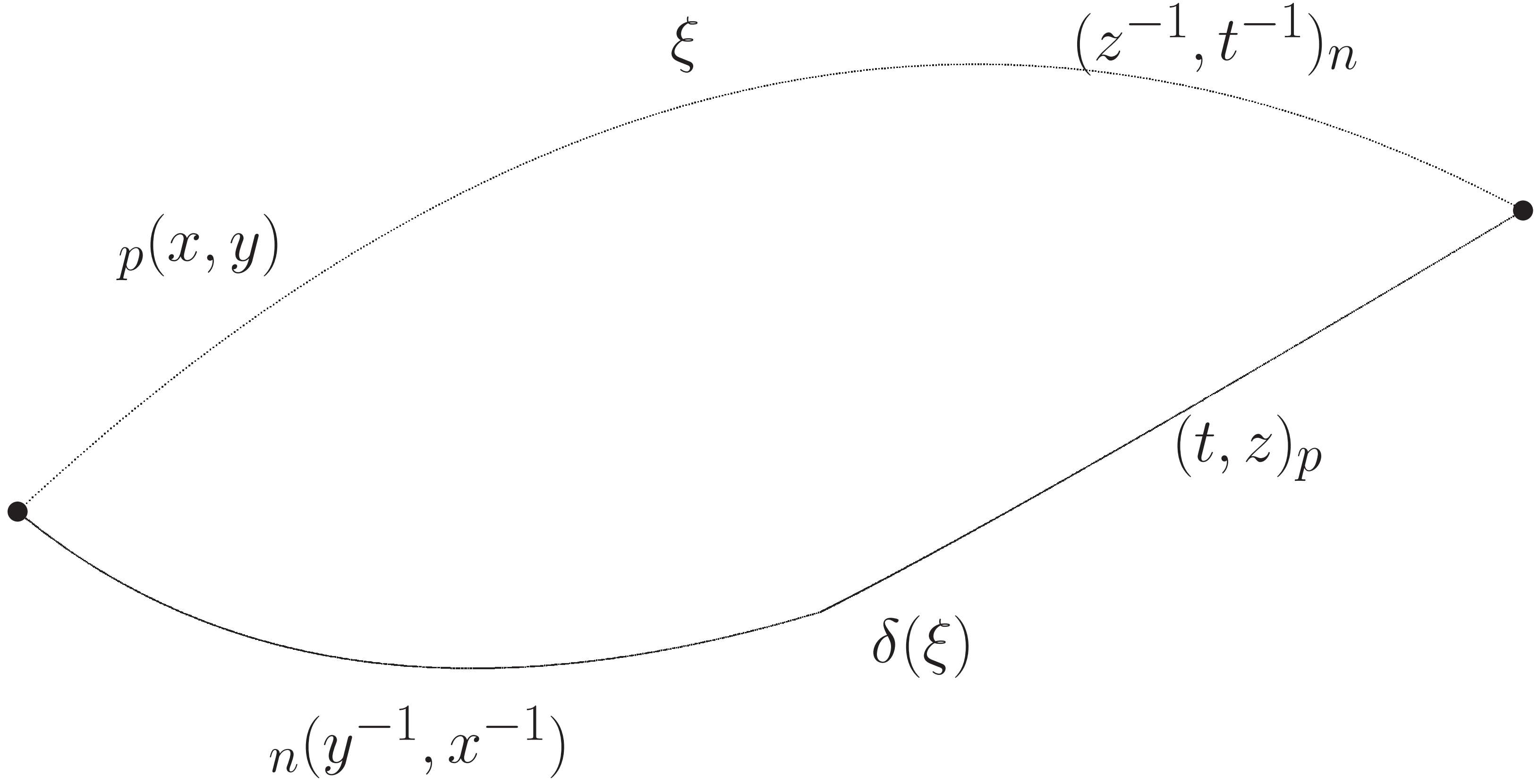
$$_p(x, y)\xi(z^{-1}, t^{-1})_n) \leftrightarrow^{\tau} {}_n(y^{-1}, x^{-1})\delta(\xi)(t, z)_p.$$

The words w and $\tau(w)$ begin and end with different generators, and are distinct geodesic reps of the same element; $w =_G \tau(w)$ follows easily from the three equations

$$_p(x, y) =_G {}_n(y^{-1}, x^{-1})\Delta, \qquad \Delta\xi =_G \delta(\xi)\Delta, \quad \Delta(z^{-1}, t^{-1})_n =_G (t, z)_p$$

We call the application of τ to a critical subword of w a τ-move on w.

Two critical words related by a τ-move.



Example:

In $G = DA_3$, aba^{-1} is critical, and $\tau(aba^{-1}) = b^{-1}ab$. So $aba^{-1} =_G b^{-1}ab$.

Now when we apply that equation (2 but not 3 times) to the word w above, we see that

$$w = (aba^{-1})b(aba^{-1})bab(aba^{-1})b^{-1} =_G (aba^{-1})b(b^{-1}ab)bab(b^{-1}ab)b^{-1},$$

and the final word freely reduces to $abbbaa$, which is geodesic.

It is straightforward to derive the following from Mathéus and Mairesse's criterion for geodesics.

Theorem (Holt, Rees, 2010)

If w is freely reduced over $\{a, b\}$ then w is shortlex minimal in DA_m unless it can be written as $w_1w_2w_3$ where w_2 is critical, and $w' = w_1\tau(w_2)w_3$ is either less than w lexicographically or not freely reduced.

Applying τ moves in Artin groups of large type.

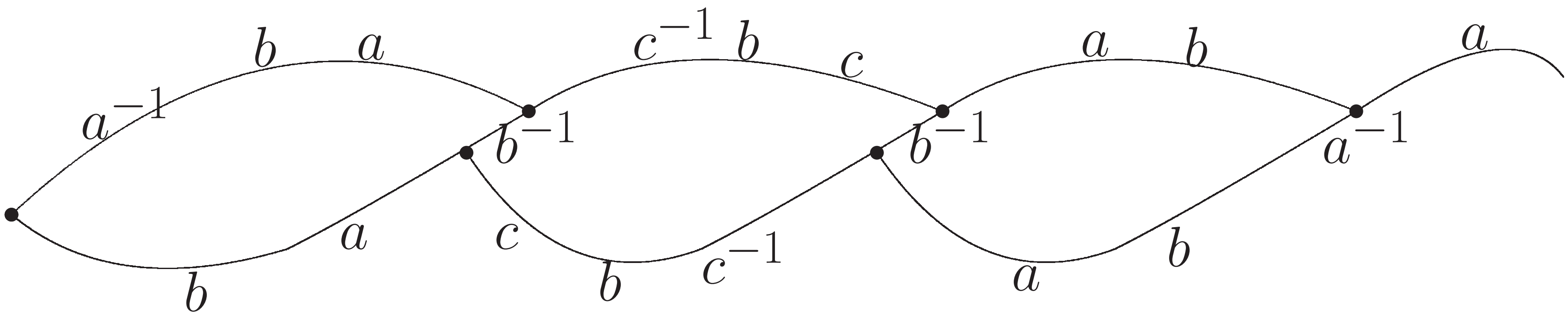
When we have more than 2 generators, we reduce to shortlex minimal form using sequences of τ-moves.

Example:

$$G = \langle a, b, c \mid aba = bab,\ aca = cac,\ bcbc = cbcb \rangle$$

First consider $w = a^{-1}bac^{-1}bcaba$. The 2 generator subwords $a^{-1}ba$, $c^{-1}bc$, aba are all geodesic in the dihedral Artin subgroups (in fact also in G). The two maximal a, b subwords are critical in DA_3. Applying a τ-move to the leftmost critical subword creates a new critical subword, to which we can then apply a τ-move.

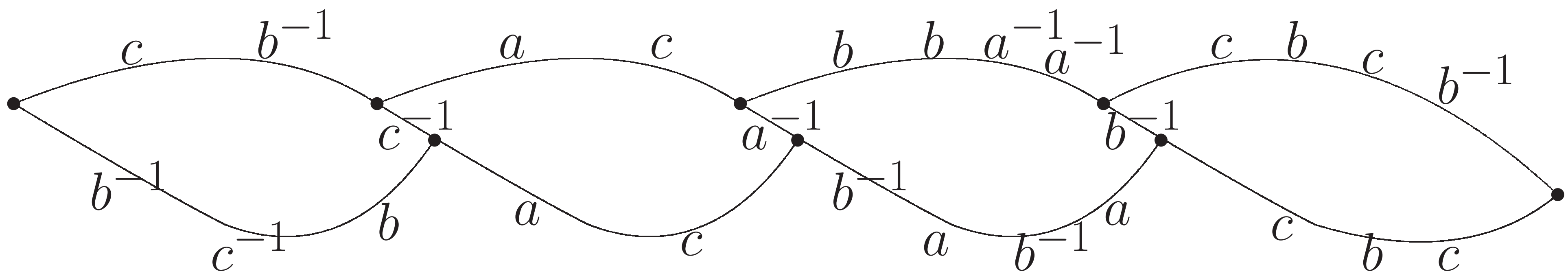
In fact, a sequence of 3 τ-moves transforms w to a word that is not freely reduced. The free reduction is then $bacbc^{-1}ab$,



Reducing $a^{-1}bac^{-1}bcaba$.

We call a sequence of τ-moves like this a **rightward length reducing sequence.**

Now consider $w = cb^{-1}acbba^{-1}a^{-1}cbcb^{-1}$, in which $bcbc^{-1}$ is critical. Applying a τ-move to the rightmost critical subword creates a new critical subword, to which we can then apply a further τ-move. A sequence of 3 τ-moves transforms w to the word $w' = b^{-1}c^{-1}bacb^{-1}ab^{-1}acbc$, of the same length as w but preceding w lexicographically.



We call a sequence like this a **leftward lex reducing sequence.**

A shortlex automatic structure

Let L be the regular set of words that excludes w iff it admits **either** a rightward length reducing sequence of τ-moves **or** a leftward lex areducing sequence of τ-moves. Certainly L contains all shortlex minimal reps. To prove our theorem we need the following.

Proposition Holt, Rees

If $w \in L$ but $wg \notin L$ then

either a single rightward sequence of τ-moves on w transforms w to a word $w'g^{-1}$, and $w' \in L$

or a single leftward sequence of τ-moves on wg transforms wg to a word w'' less than wg, and $w'' \in L$.

NB: w and w', w'' k=fellow travel, for $k = 2\max\{\mathsf{m_{ij}} : \mathsf{m_{ij}} < \infty\}$.

Crucial to the proof of the proposition:

Application of a single sequence of τ-moves to a word preserves the sequence of pairs of generators that appear in successive, overlapping, maximal 2-generator subwords.

We can check that this is valid for the reductions

$$a^{-1}bac^{-1}bcaba \to bacbc^{-1}ab \quad \text{and}$$
$$w = cb^{-1}acbba^{-1}a^{-1}cbcb^{-1} \to b^{-1}c^{-1}bacb^{-1}ab^{-1}acbc$$

in

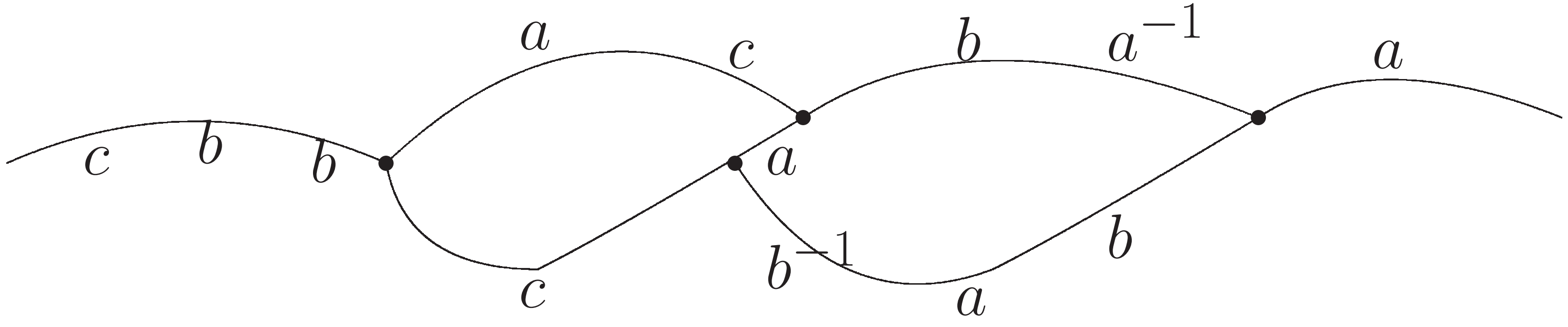
$$G = \langle a, b, c \mid aba = bab,\ aca = cac,\ bcbc = cbcb \rangle.$$

But this is a consequence of large type.

The proposition fails without large type

$$\text{Let} \quad G = \langle a, b, c \mid aba = bab,\ ac = ca,\ bcb = cbc \rangle.$$

Then $w = cbbacba^{-1} \in W$, but wb^{-1} admits a rightward length reducing sequence to $w' = cbbcb^{-1}a$, which then reduces lexicographically to $w'' = b^{-1}cbbca$. And $w''b <_{\text{slex}} w$.



In the rightward reduction of wb^{-1} a sequence of 5 overlapping max 2-gen subwords collapses to a sequence of 2.

Applying the proposition to prove G shortlex automatic

We define a map $\rho : X^{\pm *} \to L$ as follows:-

$\rho(w) := w$ if $w \in L$.

For $w \in L, wg \notin L$, wg freely reduced, $\rho(wg) := w'$ in the first case above, w'' otherwise. In this case, w and $\rho(wg)$ M-fellow travel.

More generally, for $g \in X^{\pm 1}$, v, vg freely reduced, we define $\rho(v') := \rho(v)$ if v' freely reduces to v, $\rho(vg) := \rho(\rho(v)g)$

We prove that for any $w \in W$, any generator g, $\rho(wgg^{-1}) = w$, and that $\rho(w(x_i, x_j)_{\mathsf{m_{ij}}}) = \rho(w(x_j, x_i)_{\mathsf{m_{ij}}})$; using this we can verify that whenever $w \in W$ and $v =_G w$, $\rho(v) = \rho(w)$.

So L contains a unique representative for each element of G, consists **only** of shortlex minimal reps. Fellow traveller property was noted above.

Regularity of the set of all geodesics

Suppose that G is an Artin group of large type defined over its standard generating set X.

Theorem (Holt, Rees, 2010)

The set of all geodesic words over X is regular.

By Neumann and Shapiro, this follows from the following.

Theorem (Holt, Rees, 2010)

G satisfies the Falsification by Fellow Traveller Property (FFTP), that is, $\exists k$ such that, for any non-geodesic word v over X,

$$\exists u, \quad |u| < |v|, \quad u =_G v, \quad u \sim_k v$$

Verifying FFTP

It is enough to verify FFTP for minimally non-geodesic words vg, and trivial unless vg is freely reduced, that is $\mathrm{l}[v] \neq g^{-1}$. For such a word vg, v' is a geodesic rep for vg iff $v'g^{-1}$ is a geodesic rep for v.

So the FFTP follows from the following.

Proposition (Holt, Rees)

$\exists k$ such that if v, w are geodesics in G with $\mathrm{l}[v] \neq \mathrm{l}[w]$, and $v =_G w$, then

$$\exists w', \quad w' =_G v, \quad \mathrm{l}[w'] = \mathrm{l}[w], \quad w' \sim_k v.$$

Proof: We examine the reduction of v to shortlex minimal form $\rho(v)$.