Cayley graph expanders, buildings and Beauville structures

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Oxford
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Outline

Part 1: What are expanders?

Very recommendable survey article:

S. Hoory, N. Linial, A. Wigderson: Expander graphs and their applications, *Bulletin of the American Mathematical Society* 43, 2006, pp. 439-561.

(Discrete) Cheeger constant and regular tree

Assume that $X =$ graph G with $V =$ *vertex set* and $E =$ *set of edges*. Then

$$h(G) = \inf_{\text{finite } A \subset V} \frac{|\partial A|}{\min(|A|, |V \backslash A|)},$$

where $\partial A =$ all edges connecting a vertex of A with a vertex of $V \backslash A$.

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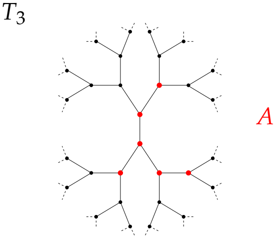
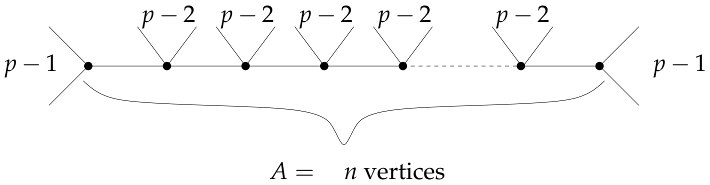
Example: $G = T_p$ (p-regular infinite tree).



Cheeger constant of regular graphs

We have $h(T_p) = p - 2$. Explanation, why $h(T_p) \leq p - 2$:



Then

$$\frac{|\partial A|}{|A|} = \frac{(n-2)(p-2) + 2(p-1)}{n} \xrightarrow[n\to\infty]{} p - 2.$$

From Cheeger constant to expanders

Properties of the discrete Cheeger constant:

- Two graphs G, G' with $V = V'$ and $E \subset E'$. Then $h(G) \leq h(G')$ (more edges/higher connectivity yields larger Cheeger constant...)

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But,...at the same time the number of edges should not increase too fast (linearly with the number of vertices).

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- $h(G_n) \geq C > 0$ *for all* n

Existence and explicit construction of expanders

- *Existence of expander families*: Using counting arguments, it can be shown for $k \geq 5$ that, for large enough n, most k-regular graphs with n vertices have Cheeger constant $\geq \frac{1}{2}$ (Pinsker 1973).

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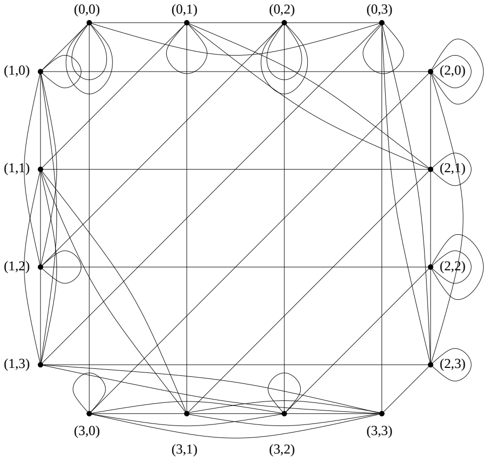
Then every $v \in V_m$ is connected to $T_1 v$, $T_2 v$, $T_1 v + e_1$, $T_2 v + e_2$.

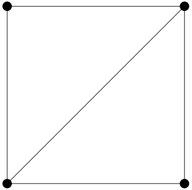
Margulis' example (1973)

The graph $G_4 = (\mathbb{Z}_4 \times \mathbb{Z}_4, E_4)$. Recall $v \rightsquigarrow T_1 v, T_2 v, T_1 v + e_1, T_2 v + e_2$.



Part 2: Spectral properties of expanders

Adjacency matrix of a finite graph $G = (V, E)$:



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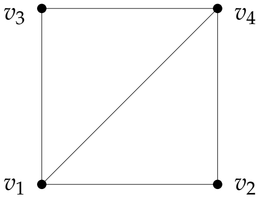

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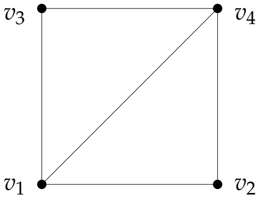

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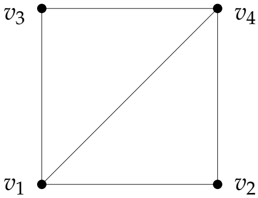

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$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Spectrum and spectral gap

Spectrum of a graph G:

$$\begin{aligned} \sigma(G) &= \{\text{eigenvalues of } A_G\}, \\ &= \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \end{aligned}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (counting with multiplicities).

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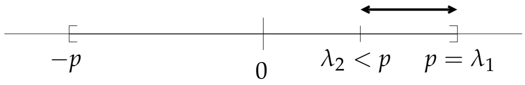
Definition

The spectral gap *of a connected, p-regular finite graph G is given by* $p - \lambda_2(G)$.



Spectral description of expanders

Theorem (Dodziuk '84, Alon-Milman '85, Alon '86)

$G = (V, E)$ finite, connected p-regular graph. Then

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Gabber-Galil showed 1981 for Margulis' example that

$$\lambda_2(G_m) \leq 5\sqrt{2} < 8,$$

so spectral gaps of all Margulis graphs are at least $8 - 5\sqrt{2}$.

Part 3: New examples of expander families

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- Lubotzky-Samuels-Vishne , Sarveniazi : expanders and Euclidean buildings

In this talk: *new families of expanders*
(joint work with N.Peyerimhoff and with R.Blok and C.Hoffman)

Theorem 1 (N.P., A.V.)

The groups

$$\Gamma_n = \langle x, y \mid r_1, r_2, r_3, [y, \underbrace{x, \ldots, x}_{n}] \rangle$$

with

$$\begin{aligned} r_1 &= yxyxyxy^{-3}x^{-3}, \\ r_2 &= yx^{-1}y^{-1}x^{-3}y^2x^{-1}yxy, \\ r_3 &= y^3x^{-1}yxyx^2y^2xyx \end{aligned}$$

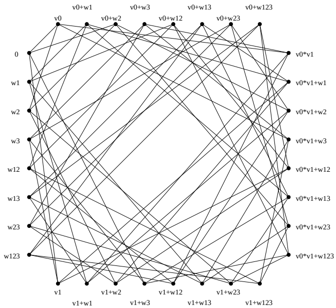
are finite, satisfy $|\Gamma_n| \to \infty$*, and the associated Cayley graphs* $Cay(\Gamma_n, \{x, y\})$ *are an expander family of 4-regular graphs.*

Nice properties: the underlying groups have only two generators and four relations.

Theorem 2 (N.P., A.V.)

There is a family of finite nilpotent groups N_k, generated by two generators $x_k.y_k$, such that $|N_k| = 2^{n_k}$ with strictly increasing $n_k \to \infty$. The Cayley graphs $G_k = Cay(N_k, \{x_k, y_k\})$ are a family of 4-regular expanders forming a tower of coverings

$$\cdots \to G_3 \to G_2 \to G_1 \to G_0.$$



(The graph G_2)

Theorem 3 (N.P., A.V.)

The pro-2 completion of the group

$$\Gamma_0 = \langle x, y \mid r_1, r_2, r_3 \rangle$$

satisfies the Golod-Shafarevich inequality

$$|R| \geq \frac{|S|^2}{4},$$

is infinite, not 2-adic analytic, contains a free subgroup of rank two but not a free pro-2 subgroup.

The construction

We start with the infinite group

$$\Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0x_1x_3, x_1x_2x_4, \ldots, x_6x_0x_2 \rangle.$$

Let $S = \{x_0, x_1, \ldots x_6\}$. Then $Cay(\Gamma, S)$ is a one-skeleton of a thick Euclidean building with the following properties:

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Γ belongs to a family considered by Edjvet,Howie 1989, and with relation to buildings by Cartwright/Steger/Mantero/Zappa 1993.

On SQ-universality of small cancellation groups

The *star graph* of $\mathcal{P}$ is the graph Γ with vertex set the disjoint union $X \cup X^{-1}$, and an edge from x to y for each *distinct* word $x^{-1}yw$ that is a cyclic permutation of an element of $R \cup R^{-1}$. The edges corresponding to $x^{-1}yw$ and $y^{-1}xw^{-1}$ form an inverse pair. Note that a relator of the form x^3, say, gives rise to exactly one inverse pair of edges between x and x^{-1}, and that each inverse pair of edges contributes a single directed edge in Γ.

On SQ-universality of small cancellation groups

Definition 1. A presentation $\mathcal{P} = \langle X \mid R \rangle$ is defined to be (m, k)-*special* for positive integers $m \geqslant 2$ and $k \geqslant 3$ if it is a finite presentation such that the following conditions hold:

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(iii) if $m = 2$ then $k \geqslant 4$.

On SQ-universality of small cancellation groups

The group G is defined to have a *special presentation* if G can be defined by an (m,k)-special presentation for some m and k. (We remark that our definition is inspired by Jim Howie whose definition of special presentation coincides with being $(3,3)$-special.)

On SQ-universality of small cancellation groups

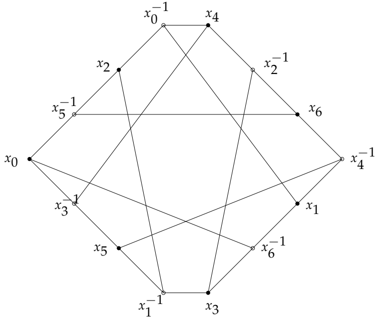
Recall that a countable group G is *SQ-universal* if every countable group can be embedded in a quotient of G.

Question (Jim Howie, 1989) Are groups with $(3,3)$-special presentation are SQ-universal?

The answer is negative (Martin Edjvet, AV)

Link and Kazdhan property (T)

The link of the vertex $e \in \Gamma$:



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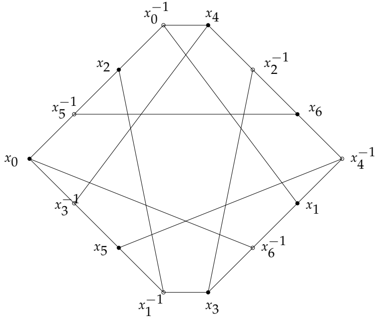

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Żuk 1996, Ballmann/Świątkowski 1997 $\Rightarrow \Gamma$ has Kazhdan property (T)

(Cartwright/Młotkowski/Steger 1993: Γ has Kazhdan property (T))

Kazhdan property (T)

Definition

A locally compact group Γ has Kazhdan property (T) if any unitary representation of Γ which has almost invariant vectors has an invariant unit vector.

Explanation: A unitary representation $\rho : \Gamma \to U(\mathcal{H})$ (with $\mathcal{H}$ a Hilbert space over $\mathbb{C}$) has almost invariant vectors if for any compact $K \subset \Gamma$ and $\epsilon > 0$ there is a unit vector $v \in \mathcal{H}$ such that $\|v - \rho(g)v\| < \epsilon$ for all $g \in K$.

Monograph:
Bekka, de la Harpe, Valette: Kazhdan's Property (T), Cambridge 2008

The construction

The subgroup of Γ, generated by x_0, x_1, is an index two subgroup given abstractly by

$$\Gamma_0 = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

where

$$\begin{aligned}
r_1 &= x_1 x_0 x_1 x_0 x_1 x_0 x_1^{-3} x_0^{-3}, \\
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\end{aligned}$$

The construction

The subgroup of Γ, generated by x_0, x_1, is an index two subgroup given abstractly by

$$\Gamma_0 = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

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We also use an explicit representation of the group Γ_0 by

$$x_0 = A_0 + A_1 \frac{1}{t}, \quad x_1 = B_0 + B_1 \frac{1}{t},$$

where A_0, A_1 are certain 9×9 matrices over $\mathbb{F}_2$.

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Rewriting this representation with finite band Toeplitz matrices and establishing certain periodicity patterns for higher commutators are at the heart of the proofs of Theorems 1 and 2.

We have $x_0 = A_0 + \frac{1}{y}A_1$ and $x_1 = B_0 + \frac{1}{y}B_1$ with 9×9 matrices $A_0, A_1, B_0, B_1 \in \mathrm{M}(9, \mathbb{F}_2)$, and their inverses x_0^{-1}, x_1^{-1} are of the same form. Therefore, an arbitrary group element $x \in G$ is of the form

$$x = C_0 + \sum_{j=1}^{k} \frac{1}{y^j} C_j,$$

which we identify with the (finite band) upper triangular infinite Toeplitz matrix

$$x = \begin{pmatrix} C_0 & C_1 & C_2 & \dots & C_k & 0 & 0 & \dots \\ 0 & C_0 & C_1 & \dots & C_{k-1} & C_k & 0 & \ddots \\ 0 & 0 & C_0 & \dots & C_{k-2} & C_{k-1} & C_k & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

where each C_i is a matrix in $\mathrm{M}(9, \mathbb{F}_2)$. One checks that multiplication of elements in $\mathrm{GL}(9, \mathbb{F}_2[1/y])$ and of the corresponding infinite matrices is consistent.

A conjecture

MAGMA calculations support the following

Conjecture (N.P., A.V.)

Let $\lambda_n(\Gamma_0)$ be the lower exponent-2 central series of Γ_0, i.e.,

$$\lambda_{n+1} = [\lambda_n, \Gamma_0]\lambda_n^2.$$

Then we have the following 3-periodicity for the abelian quotients λ_n/λ_{n+1}, $n \geq 2$:

$$|\lambda_n/\lambda_{n+1}| = \begin{cases} 8, & \text{if } n \equiv 0, 1 \bmod 3, \\ 4, & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

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This conjecture would imply that the tower of coverings of expander graphs in Theorem 2 would always have very low covering indices $4, 8, 4, 4, 8, 4, 4, 8, 4, 4, 8, 4, 4, \ldots$.

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New girth and finite simple groups

Definition

Let $X = (V, E)$ be a finite k-regular graph with n vertices. we say that X is an (n, k, c) expander if for any subset $A \subset V$

$$|\partial A| \geq c(1 - \frac{|A|}{N})|A|$$

where $\partial A = \{v \in V \mid d(v, A) = 1\}$.

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Theorem (Margulis)

Let Γ be a finitely generated group that has property (T). Let $\mathcal{L}$ be a family of finite index normal subgroups of Γ and let $S = S^{-1}$ be a finite symmetric set of generators for γ. Then the family $\{X(\Gamma/N, S) \mid N \in \mathcal{L}\}$ of Cayley graphs of the finite quotients of Γ with respect to the image of S is a family of (n, k, c) expanders for $n = |\Gamma/N|, k = |S|$ and some fixed $c > 0$.

New girth and finite simple groups

Definition

Let $\Gamma, \mathcal{L}, S$ as above. Consider also the natural map $\phi_N : X(\Gamma, S) \rightarrow X(\Gamma/N, S)$. The *new girth* of a graph $X(\Gamma/N, S)$ is the length of the shortest circuit γ in $X(\Gamma/N, S)$ so that γ is not the image of a circuit in $X(\Gamma, S)$ under the map ϕ_N.

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Theorem (R.Blok,C.Hoffman,AV)

For any n there exists an $\epsilon > 0$ and a symmetric set $S_{n,q}$ of generators for $SU_{2n}(q)$ so that $S_{n,q}$ has size five and the family of Cayley graphs $X(SU_{2n}(q), S_{n,q})$ for $q \geq n$ forms an ϵ-expanding family of unbounded new girth.

Beauville surfaces

Definition

A Beauville surface S is a complex algebraic surface, of the form $(C_1 \times C_2)/G$, where C_1 and C_2 are non-singular projective curves of genera $g(C_i) \geqslant 2$, and G is a finite group acting freely on the product of curves by holomorphic transformations.

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Definition

A surface S is said to be *isogenous to a product* is S admits finite unramified covering isomorphic to a product of curves.

Beauville surfaces

Definition

Let S be a surface isogenous to a product with minimal realisation $S \cong (C_1 \times C_2)/G$. We say that S is a *mixed case* if the action of G exchanges the two factors (and then C_1 and C_2 are isomorphic) and an *unmixed case* if G acts via a diagonal action.

Beauville surfaces

Beauville's original example had two curves $C_1 = C_2$, given by the Fermat curve $x^5 + y^5 + z^5 = 0$, and G the group $(\mathbb{Z}/5\mathbb{Z})^2$ acting on $C_1 \times C_2$ by the rule

$$(a, b) \cdot ([x : y : z], [u : v : w]) = ([\xi^a x : \xi^b y : z], [\xi^{a+3b} u : \xi^{2a+4b} v : w]),$$

where $\xi = e^{\frac{2\pi i}{5}}$ and $a, b \in \mathbb{Z}/5\mathbb{Z}$. Then S is a Beauville surface of unmixed type with $g(C_1) = g(C_2) = 6$.

Beauville surfaces

Let G be a finite group and r an integer with $r \geqslant 2$. An r-tuple $T = [g_1, ..., g_r]$ of elements of G is called a *spherical system of generators*, if $g_1, ..., g_r$ generate G and we additionally have $g_1 ... g_r = 1$.
For an r-tuple $T = [g_1, ..., g_r]$ of elements of G and $g \in G$, we set

$$gTg^{-1} := [gg_1g^{-1}, ..., gg_rg^{-1}].$$

If $A = [m_1, ..., m_r]$ is an r-tuple of natural numbers with $2 \leqslant m_1 \leqslant ... \leqslant m_r$, then the spherical system of generators $T = [g_1, ..., g_r]$ is said to be *of type* A, if there is a permutation $\tau \in Sym(r)$ such that we have

$$ord(g_1) = m_{\tau(1)}, ord(g_2) = m_{\tau(2)}, \cdots, ord(g_r) = m_{\tau(r)}.$$

(Here $ord(g)$ is the order of the element $g \in G$.)

Beauville surfaces

For a spherical system of generators $T = [g_1, ..., g_r]$ of G, we define

$$\Sigma(T) = \Sigma([g_1, ..., g_r]) := \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \bigcup_{i=1}^{r} \{g \cdot g_i^j \cdot g^{-1}\} \qquad (2)$$

to be the union of all conjugates of the cyclic subgroups generated by the elements $g_1, ..., g_r$. A pair of spherical systems of generators (T_1, T_2) of G is called *disjoint* if

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\}.$$

Beauville surfaces

Next, we introduce unmixed and mixed ramification structures.

Definition

Let $A_1 = [m_{(1,1)}, .., m_{(1,r)}]$ and $A_2 = [m_{(2,1)}, .., m_{(2,s)}]$ be tuples of natural numbers with $2 \leqslant m_{(1,1)} \leqslant ... \leqslant m_{(1,r)}$ and $2 \leqslant m_{(2,1)} \leqslant ... \leqslant m_{(2,s)}$. An *unmixed ramification structure of type* (A_1, A_2) *for* G is a disjoint pair (T_1, T_2) of spherical systems of generators, such that T_1 has type A_1 and T_2 has type A_2.

The disjointness of the pair (T_1, T_2) in the definition of an unmixed ramification structure guarantees that G acts freely on the product $C_{T_1} \times C_{T_2}$ of the associated algebraic curves

Beauville surfaces

Definition

Let $A = [m_1, ..., m_r]$ be an r-tuple of natural numbers with $2 \leqslant m_1 \leqslant ... \leqslant m_r$. A *mixed ramification structure of type A for G* is a pair (H, T) where H is a subgroup of index 2 in G and $T = [g_1, ...g_r]$ is an r-tuple of elements of G such that the following hold:

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- for every $g \in G \setminus H$, the spherical systems T and gTg^{-1} are disjoint,

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- T is a spherical system of generators of H of type A,
- for every $g \in G \setminus H$, the spherical systems T and gTg^{-1} are disjoint,
- for every $g \in G \setminus H$ we have $g^2 \notin \Sigma(T)$.

Beauville surfaces

Definition

An *unmixed Beauville structure* is an unmixed ramification structure with two spherical systems (T_1, T_2) of generators of length 3, i.e., $r = 3$ and $s = 3$.
A *mixed Beauville structure* is a mixed ramification structure with a spherical system T of generators of length 3, i.e., $r = 3$.

Beauville surfaces

Let G_1, G_2,... be the lower 2-central series for the group Γ_0 and H_1,H_2,... be the lower 2-central series for Γ.

Theorem (N.Barker,N.Boston,N.Peyerimhoff,AV)

Let $3 \leq k \leq 64$. If k is not a power of 2, then Γ / H_k admits an unmixed Beauville structure.

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Let $3 \leq k \leq 10$. If k is not a power of 2, then Γ_0 / G_k admits a mixed Beauville structure.