Cayley graph expanders, buildings and Beauville structures

A. Vdovina
Newcastle University

Oxford
11 November 2010
Outline

What are expanders?

Spectral properties of expanders

New examples of expander families (joint work with N.Peyerimhoff)

On SQ-universality of small cancellation groups (joint work with Martin Edjvet)

More about methods

New girth and finite simple groups (joint work with R.Blok and C.Hoffman)

Beauville surfaces (joint work with N.Barker, N.Boston, N.Peyerimhoff)
Part 1: What are expanders?
Very recommendable survey article:
S. Hoory, N. Linial, A. Wigderson: Expander graphs and their applications, 
(Discrete) Cheeger constant and regular tree

Assume that $X = \text{graph } G$ with $V = \text{vertex set}$ and $E = \text{set of edges}$. Then

$$h(G) = \inf_{\text{finite } A \subseteq V} \frac{|\partial A|}{\min(|A|, |V \setminus A|)},$$

where $\partial A = \text{all edges connecting a vertex of } A \text{ with a vertex of } V \setminus A.$
(Discrete) Cheeger constant and regular tree

Assume that $X = \text{graph } G$ with $V = \text{vertex set}$ and $E = \text{set of edges}$. Then

$$h(G) = \inf_{\text{finite } A \subset V} \frac{|\partial A|}{\min(|A|, |V \setminus A|)},$$

where $\partial A$ = all edges connecting a vertex of $A$ with a vertex of $V \setminus A$.

**Example:** $G = T_p$ ($p$-regular infinite tree).

$$T_3$$

A
Cheeger constant of regular graphs

We have $h(T_p) = p - 2$. Explanation, why $h(T_p) \leq p - 2$:

$$h(T_p) = p - 2$$

Then

$$\frac{|\partial A|}{|A|} = \frac{(n-2)(p-2) + 2(p-1)}{n} \xrightarrow{n \to \infty} p - 2.$$
From Cheeger constant to expanders

Properties of the discrete Cheeger constant:

- Two graphs $G, G'$ with $V = V'$ and $E \subset E'$. Then $h(G) \leq h(G')$ (more edges/higher connectivity yields larger Cheeger constant...)

A. Vdovina
Newcastle University
Cayley graph expanders, buildings and Beauville structures
From Cheeger constant to expanders

Properties of the discrete Cheeger constant:

- Two graphs $G, G'$ with $V = V'$ and $E \subset E'$. Then $h(G) \leq h(G')$ (more edges/higher connectivity yields larger Cheeger constant...)

- $\alpha = h(G)$ can be interpreted as edge expansion rate: given any set $A$ of $k$ vertices, then there are at least $\alpha k$ edges connecting them with vertices in the complement.
From Cheeger constant to expanders

Properties of the discrete Cheeger constant:

- Two graphs $G, G'$ with $V = V'$ and $E \subset E'$. Then $h(G) \leq h(G')$ (more edges/higher connectivity yields larger Cheeger constant...)

- $\alpha = h(G)$ can be interpreted as *edge expansion rate*: given any set $A$ of $k$ vertices, then there are at least $\alpha k$ edges connecting them with vertices in the complement.

We are looking for families of increasing finite graphs $G_n$ with uniform lower bound on their Cheeger constants:

$$h(G_n) \geq C > 0.$$  

But,...
From Cheeger constant to expanders

Properties of the discrete Cheeger constant:

- Two graphs $G, G'$ with $V = V'$ and $E \subset E'$. Then $h(G) \leq h(G')$ (more edges/higher connectivity yields larger Cheeger constant...)

- $\alpha = h(G)$ can be interpreted as edge expansion rate: given any set $A$ of $k$ vertices, then there are at least $\alpha k$ edges connecting them with vertices in the complement.

We are looking for families of increasing finite graphs $G_n$ with uniform lower bound on their Cheeger constants:

$$h(G_n) \geq C > 0.$$ 

But,...at the same time the number of edges should not increase too fast (linearly with the number of vertices).

A. Vdovina
Newcastle University

Cayley graph expanders, buildings and Beauville structures
Definition of expander graphs

**Definition**

A sequence $G_n = (V_n, E_n)$ of connected finite graphs with $|V_n| \to \infty$ is called a family of expanders if
Definition of expander graphs

Definition
A sequence $G_n = (V_n, E_n)$ of connected finite graphs with $|V_n| \to \infty$ is called a family of expanders if

- all $G_n$ are $p$-regular graphs ($\Rightarrow |E_n| = \frac{p}{2} |V_n|$)
Definition of expander graphs

Definition

A sequence $G_n = (V_n, E_n)$ of connected finite graphs with $|V_n| \to \infty$ is called a family of expanders if

- all $G_n$ are $p$-regular graphs ($\Rightarrow |E_n| = \frac{p}{2} |V_n|$)
- $h(G_n) \geq C > 0$ for all $n$
Existence and explicit construction of expanders

- **Existence of expander families:** Using counting arguments, it can be shown for $k \geq 5$ that, for large enough $n$, most $k$-regular graphs with $n$ vertices have Cheeger constant $\geq \frac{1}{2}$ (Pinsker 1973).
Existence and explicit construction of expanders

- **Existence of expander families**: Using counting arguments, it can be shown for $k \geq 5$ that, for large enough $n$, most $k$-regular graphs with $n$ vertices have Cheeger constant $\geq \frac{1}{2}$ (Pinsker 1973).
- **Explicit construction** of expanders is difficult.
Existence and explicit construction of expanders

- **Existence of expander families**: Using counting arguments, it can be shown for $k \geq 5$ that, for large enough $n$, most $k$-regular graphs with $n$ vertices have Cheeger constant $\geq \frac{1}{2}$ (Pinsker 1973).
- **Explicit construction** of expanders is difficult.

First explicit construction was given by Margulis 1973 (based on Kazhdan’s property (T)): 
Existence and explicit construction of expanders

- **Existence of expander families**: Using counting arguments, it can be shown for \( k \geq 5 \) that, for large enough \( n \), most \( k \)-regular graphs with \( n \) vertices have Cheeger constant \( \geq \frac{1}{2} \) (Pinsker 1973).

- **Explicit construction** of expanders is difficult.

First explicit construction was given by Margulis 1973 (based on Kazhdan’s property (T)):

The graphs \( G_m = (V_m, E_m) \) are 8-regular graphs with vertex set

\[ V_m = \mathbb{Z}_m \times \mathbb{Z}_m. \]
Existence and explicit construction of expanders

- **Existence of expander families**: Using counting arguments, it can be shown for $k \geq 5$ that, for large enough $n$, most $k$-regular graphs with $n$ vertices have Cheeger constant $\geq \frac{1}{2}$ (Pinsker 1973).

- **Explicit construction** of expanders is difficult.

First explicit construction was given by Margulis 1973 (based on Kazhdan’s property (T)):

The graphs $G_m = (V_m, E_m)$ are 8-regular graphs with vertex set

$$V_m = \mathbb{Z}_m \times \mathbb{Z}_m.$$ 

Let

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. $$
Existence and explicit construction of expanders

- **Existence of expander families**: Using counting arguments, it can be shown for $k \geq 5$ that, for large enough $n$, most $k$-regular graphs with $n$ vertices have Cheeger constant $\geq \frac{1}{2}$ (Pinsker 1973).

- **Explicit construction** of expanders is difficult.

First explicit construction was given by Margulis 1973 (based on Kazhdan’s property (T)):

The graphs $G_m = (V_m, E_m)$ are 8-regular graphs with vertex set

$$V_m = \mathbb{Z}_m \times \mathbb{Z}_m.$$ 

Let

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Then every $v \in V_m$ is connected to $T_1v$, $T_2v$, $T_1v + e_1$, $T_2v + e_2$. 
Margulis’ example (1973)

The graph $G_4 = (\mathbb{Z}_4 \times \mathbb{Z}_4, E_4)$. Recall $v \sim T_1 v, T_2 v, T_1 v + e_1, T_2 v + e_2$.
Part 2: Spectral properties of expanders

Adjacency matrix of a finite graph $G = (V, E)$:
Part 2: Spectral properties of expanders

Adjacency matrix of a finite graph $G = (V, E)$:

Label the vertices as $v_1, \ldots, v_n$. 
Part 2: Spectral properties of expanders

Adjacency matrix of a finite graph $G = (V, E)$:

Label the vertices as $v_1, \ldots, v_n$. Define symmetric $n \times n$ matrix $A_G = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases}$$
Part 2: Spectral properties of expanders

Adjacency matrix of a finite graph $G = (V, E)$:

Label the vertices as $v_1, \ldots, v_n$. Define symmetric $n \times n$ matrix $A_G = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise}. \end{cases}$$

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$
Spectrum and spectral gap

Spectrum of a graph $G$:

\[ \sigma(G) = \{ \text{eigenvalues of } A_G \}, \]
\[ = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \]

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (counting with multiplicities).
Spectrum and spectral gap

Spectrum of a graph $G$:

$$\sigma(G) = \{\text{eigenvalues of } A_G\},$$
$$= \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (counting with multiplicities).

Facts:

- If $G$ is $p$-regular then $\lambda_1 = p$ and $\lambda_1, \ldots, \lambda_n \in [-p, p]$. In this case:
What are expanders?

Spectral properties of expanders

New examples of expander families (joint work with N. Peyerimhoff)

On SQ-universality of small cancellation

Cayley graph expanders, buildings and Beauville structures

Spectrum and spectral gap

Spectrum of a graph $G$:

$$
\sigma(G) = \{\text{eigenvalues of } A_G\},
\lambda_1, \lambda_2, \ldots, \lambda_n
$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (counting with multiplicities).

Facts:

- If $G$ is $p$-regular then $\lambda_1 = p$ and $\lambda_1, \ldots, \lambda_n \in [-p,p]$. In this case:
  - $G$ is connected iff $\lambda_2 < p$. 

A. Vdovina
Newcastle University
Spectrum and spectral gap

Spectrum of a graph \( G \):

\[
\sigma(G) = \{\text{eigenvalues of } A_G\},
\]

\[
= \{\lambda_1, \lambda_2, \ldots, \lambda_n\}
\]

with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) (counting with multiplicities).

Facts:

- If \( G \) is \( p \)-regular then \( \lambda_1 = p \) and \( \lambda_1, \ldots, \lambda_n \in [-p, p] \). In this case:
  - \( G \) is connected iff \( \lambda_2 < p \).
  - \( G \) is bipartite iff \( \lambda_n = -p \).
**Spectrum and spectral gap**

Spectrum of a graph $G$:

$$\sigma(G) = \{\text{eigenvalues of } A_G\},$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ (counting with multiplicities).

Facts:

- If $G$ is $p$-regular then $\lambda_1 = p$ and $\lambda_1, \ldots, \lambda_n \in [-p, p]$. In this case:
  - $G$ is connected iff $\lambda_2 < p$.
  - $G$ is bipartite iff $\lambda_n = -p$.

**Definition**

The spectral gap of a connected, $p$-regular finite graph $G$ is given by $p - \lambda_2(G)$.
Spectral description of expanders

Theorem (Dodziuk ’84, Alon-Milman ’85, Alon ’86)

\[ G = (V, E) \text{ finite, connected } p\text{-regular graph. Then} \]

\[ \frac{p - \lambda_2}{2} \leq h(G) \leq \sqrt{2p(p - \lambda_2)}. \]
Spectral description of expanders

Theorem (Dodziuk ’84, Alon-Milman ’85, Alon ’86)

\( G = (V, E) \) finite, connected \( p \)-regular graph. Then

\[
\frac{p - \lambda_2}{2} \leq h(G) \leq \sqrt{2p(p - \lambda_2)}.
\]

\( \sim \) Alternative definition of expander family: Sequence of increasing \( p \)-regular graphs \( G_n \) with \( p - \lambda_2(G_n) \geq C > 0 \).
Spectral description of expanders

Theorem (Dodziuk ’84, Alon-Milman ’85, Alon ’86)

\[ G = (V, E) \text{ finite, connected } p\text{-regular graph. Then} \]
\[ \frac{p - \lambda_2}{2} \leq h(G) \leq \sqrt{2p(p - \lambda_2)}. \]

\text{Alternative definition of expander family: Sequence of increasing } \]
\text{ } p\text{-regular graphs } G_n \text{ with } p - \lambda_2(G_n) \geq C > 0.

Gabber-Galil showed 1981 for Margulis’ example that

\[ \lambda_2(G_m) \leq 5\sqrt{2} < 8, \]

so spectral gaps of all Margulis graphs are at least \(8 - 5\sqrt{2}\).
Part 3: New examples of expander families

- Lubotzky-Phillips-Sarnak, Margulis: Ramanujan Graphs
Part 3: New examples of expander families

- Lubotzky-Phillips-Sarnak, Margulis: Ramanujan Graphs
- Reingold-Vadham-Wigderson, Rozenman-Shalev-Wigderson: expanders via zig-zag product
Part 3: New examples of expander families

- Lubotzky-Phillips-Sarnak, Margulis: Ramanujan Graphs
- Reingold-Vadham-Wigderson, Rozenman-Shalev-Wigderson: expanders via zig-zag product
- Kassabov: Symmetric groups and expanders
Part 3: New examples of expander families

- Lubotzky-Phillips-Sarnak, Margulis: Ramanujan Graphs
- Reingold-Vadham-Wigderson, Rozenman-Shalev-Wigderson: expanders via zig-zag product
- Kassabov: Symmetric groups and expanders
- Kassabov- Lubotzky-Nikolov and Breuillard-Green-Tao: Finite simple groups as expanders
Part 3: New examples of expander families

- Lubotzky-Phillips-Sarnak, Margulis: Ramanujan Graphs
- Reingold-Vadham-Wigderson, Rozenman-Shalev-Wigderson: expanders via zig-zag product
- Kassabov: Symmetric groups and expanders
- Kassabov- Lubotzky-Nikolov and Breuillard-Green-Tao: Finite simple groups as expanders
- Bourgain-Gamburd: expanders and $SL_2(p)$
Part 3: New examples of expander families

- Lubotzky-Phillips-Sarnak, Margulis: Ramanujan Graphs
- Reingold-Vadham-Wigderson, Rozenman-Shalev-Wigderson: expanders via zig-zag product
- Kassabov: Symmetric groups and expanders
- Kassabov- Lubotzky-Nikolov and Breuillard-Green-Tao: Finite simple groups as expanders
- Bourgain-Gamburd: expanders and $SL_2(p)$
- Lubotzky-Samuels-Vishne, Sarveniazi: expanders and Euclidean buildings

In this talk: new families of expanders
(joint work with N.Peyerimhoff and with R.Blok and C.Hoffman)
Theorem 1 (N.P., A.V.)

The groups

\[ \Gamma_n = \langle x, y \mid r_1, r_2, r_3, [y, x, \ldots, x] \rangle \]

with

\[
\begin{align*}
    r_1 &= yxxyxy^{-3}x^{-3}, \\
    r_2 &= yx^{-1}y^{-1}x^{-3}y^2x^{-1}yxy, \\
    r_3 &= y^3x^{-1}yxyx^2y^2xyx
\end{align*}
\]

are finite, satisfy \( |\Gamma_n| \to \infty \), and the associated Cayley graphs \( \text{Cay}(\Gamma_n, \{x, y\}) \) are an expander family of 4-regular graphs.

Nice properties: the underlying groups have only two generators and four relations.
Theorem 2 (N.P., A.V.)

There is a family of finite nilpotent groups $N_k$, generated by two generators $x_k, y_k$, such that $|N_k| = 2^{n_k}$ with strictly increasing $n_k \to \infty$. The Cayley graphs $G_k = \text{Cay}(N_k, \{x_k, y_k\})$ are a family of 4-regular expanders forming a tower of coverings

$$\cdots \to G_3 \to G_2 \to G_1 \to G_0.$$
Theorem 3 (N.P., A.V.)

The pro-$2$ completion of the group

\[ \Gamma_0 = \langle x, y \mid r_1, r_2, r_3 \rangle \]

satisfies the Golod-Shafarevich inequality

\[ |R| \geq \frac{|S|^2}{4}, \]

is infinite, not $2$-adic analytic, contains a free subgroup of rank two but not a free pro-$2$ subgroup.
The construction

We start with the infinite group

\[ \Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0x_1x_3, x_1x_2x_4, \ldots, x_6x_0x_2 \rangle. \]

Let \( S = \{x_0, x_1, \ldots x_6\} \). Then \( \text{Cay}(\Gamma, S) \) is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)
The construction

We start with the infinite group

$$\Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0 x_1 x_3, x_1 x_2 x_4, \ldots, x_6 x_0 x_2 \rangle.$$ 

Let $S = \{x_0, x_1, \ldots x_6\}$. Then $\text{Cay}(\Gamma, S)$ is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)
- every vertex has degree 14
The construction

We start with the infinite group

\[ \Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0x_1x_3, x_1x_2x_4, \ldots, x_6x_0x_2 \rangle. \]

Let \( S = \{x_0, x_1, \ldots x_6\} \). Then \( \text{Cay}(\Gamma, S) \) is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)
- every vertex has degree 14
- 3 triangles meet at every edge
The construction

We start with the infinite group

\[ \Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0 x_1 x_3, x_1 x_2 x_4, \ldots, x_6 x_0 x_2 \rangle. \]

Let \( S = \{x_0, x_1, \ldots, x_6\} \). Then \( \text{Cay}(\Gamma, S) \) is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)
- every vertex has degree 14
- 3 triangles meet at every edge
- any two triangles \( \Delta_1, \Delta_2 \) lie in a common plane (apartment) tessellated by equilateral triangles
The construction

We start with the infinite group

\[ \Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0x_1x_3, x_1x_2x_4, \ldots, x_6x_0x_2 \rangle. \]

Let \( S = \{x_0, x_1, \ldots, x_6\} \). Then \( \text{Cay}(\Gamma, S) \) is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)
- every vertex has degree 14
- 3 triangles meet at every edge
- any two triangles \( \Delta_1, \Delta_2 \) lie in a common plane (apartment) tessellated by equilateral triangles
- link of every vertex is the incidence graph of a finite projective plane with 7 points
The construction

We start with the infinite group

\[ \Gamma = \langle x_0, x_1, \ldots, x_6 \mid x_0x_1x_3, x_1x_2x_4, \ldots, x_6x_0x_2 \rangle. \]

Let \( S = \{x_0, x_1, \ldots, x_6\} \). Then \( \text{Cay}(\Gamma, S) \) is a one-skeleton of a thick Euclidean building with the following properties:

- lots of (equilateral) triangles (chambers)
- every vertex has degree 14
- 3 triangles meet at every edge
- any two triangles \( \Delta_1, \Delta_2 \) lie in a common plane (apartment) tessellated by equilateral triangles
- link of every vertex is the incidence graph of a finite projective plane with 7 points

\( \Gamma \) belongs to a family considered by Edjvet, Howie 1989, and with relation to buildings by Cartwright/Steger/Mantero/Zappa 1993.
On SQ-universality of small cancellation groups

The \textit{star graph} of $\mathcal{P}$ is the graph $\Gamma$ with vertex set the disjoint union $X \cup X^{-1}$, and an edge from $x$ to $y$ for each \textit{distinct} word $x^{-1}yw$ that is a cyclic permutation of an element of $R \cup R^{-1}$. The edges corresponding to $x^{-1}yw$ and $y^{-1}xw^{-1}$ form an inverse pair. Note that a relator of the form $x^3$, say, gives rise to exactly one inverse pair of edges between $x$ and $x^{-1}$, and that each inverse pair of edges contributes a single directed edge in $\Gamma$. 
On SQ-universality of small cancellation groups

Definition 1. A presentation $\mathcal{P} = \langle X \mid R \rangle$ is defined to be $(m, k)$-special for positive integers $m \geq 2$ and $k \geq 3$ if it is a finite presentation such that the following conditions hold:

(i) the star graph $\Gamma$ of $\mathcal{P}$ is a connected, bipartite graph of diameter $m$ and girth $2m$ in which each vertex has degree at least 3;
On SQ-universality of small cancellation groups

Definition 1. A presentation $\mathcal{P} = \langle X \mid R \rangle$ is defined to be $(m, k)$-special for positive integers $m \geq 2$ and $k \geq 3$ if it is a finite presentation such that the following conditions hold:

(i) the star graph $\Gamma$ of $\mathcal{P}$ is a connected, bipartite graph of diameter $m$ and girth $2m$ in which each vertex has degree at least 3;

(ii) each relator $r \in R$ has length $k$; and
On SQ-universality of small cancellation groups

**Definition 1.** A presentation $\mathcal{P} = \langle X \mid R \rangle$ is defined to be $(m, k)$-special for positive integers $m \geq 2$ and $k \geq 3$ if it is a finite presentation such that the following conditions hold:

(i) the star graph $\Gamma$ of $\mathcal{P}$ is a connected, bipartite graph of diameter $m$ and girth $2m$ in which each vertex has degree at least 3;

(ii) each relator $r \in R$ has length $k$; and

(iii) if $m = 2$ then $k \geq 4$. 
On SQ-universality of small cancellation groups

The group $G$ is defined to have a *special presentation* if $G$ can be defined by an $(m,k)$-special presentation for some $m$ and $k$. (We remark that our definition is inspired by Jim Howie whose definition of special presentation coincides with being $(3,3)$-special.)
On SQ-universality of small cancellation groups

Recall that a countable group $G$ is *SQ-universal* if every countable group can be embedded in a quotient of $G$.

**Question (Jim Howie, 1989)** Are groups with $(3, 3)$-special presentation are SQ-universal?

The answer is negative (Martin Edjvet, AV)
Link and Kazdhan property (T)

The link of the vertex $e \in \Gamma$:

Interpreting $x_i$ as POINTS and $x_i^{-1}$ as LINES yields the incidence relations of a finite projective plane.
Link and Kazhdan property (T)

The link of the vertex $e \in \Gamma$:

Interpreting $x_i$ as POINTS and $x_i^{-1}$ as LINES yields the incidence relations of a finite projective plane.

Žuk 1996, Ballmann/Świątkowski 1997 $\Rightarrow \Gamma$ has Kazhdan property (T)  
(Cartwright/Młotkowski/Steger 1993: $\Gamma$ has Kazhdan property (T))
Kazhdan property (T)

**Definition**

A locally compact group \( \Gamma \) has Kazhdan property (T) if any unitary representation of \( \Gamma \) which has almost invariant vectors has an invariant unit vector.

**Explanation:** A unitary representation \( \rho : \Gamma \to U(\mathcal{H}) \) (with \( \mathcal{H} \) a Hilbert space over \( \mathbb{C} \)) has almost invariant vectors if for any compact \( K \subset \Gamma \) and \( \epsilon > 0 \) there is a unit vector \( v \in \mathcal{H} \) such that \( \|v - \rho(g)v\| < \epsilon \) for all \( g \in K \).

**Monograph:**
Bekka, de la Harpe, Valette: Kazhdan’s Property (T), Cambridge 2008
The construction

The subgroup of $\Gamma$, generated by $x_0, x_1$, is an index two subgroup given abstractly by

$$\Gamma_0 = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

where

$$r_1 = x_1 x_0 x_1 x_0 x_1 x_0 x_1^{-3} x_0^{-3},$$

$$r_2 = x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_1^{-1} x_0^{-1} x_1 x_0 x_1,$$

$$r_3 = x_1^3 x_0^{-1} x_1 x_0 x_1^2 x_0^2 x_0 x_1 x_0.$$
The construction

The subgroup of $\Gamma$, generated by $x_0, x_1$, is an index two subgroup given abstractly by

$$\Gamma_0 = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

where

$$r_1 = x_1 x_0 x_1 x_0 x_1 x_0 x_1 x_0^{-3} x_0^{-3},$$
$$r_2 = x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_1 x_0 x_1,$$
$$r_3 = x_1^3 x_0^{-1} x_1 x_0 x_1 x_0 x_1 x_0 x_1 x_0.$$

We also use an explicit representation of the group $\Gamma_0$ by

$$x_0 = A_0 + A_1 \frac{1}{t}, \quad x_1 = B_0 + B_1 \frac{1}{t},$$

where $A_0, A_1$ are certain $9 \times 9$ matrices over $\mathbb{F}_2$. 
The construction

The subgroup of $\Gamma$, generated by $x_0, x_1$, is an index two subgroup given abstractly by

$$\Gamma_0 = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

where

$$
\begin{align*}
    r_1 &= x_1 x_0 x_1 x_0 x_1 x_0^{-3} x_1^{-3}, \\
    r_2 &= x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_0^{-1} x_1 x_0 x_1, \\
    r_3 &= x_1^3 x_0^{-1} x_1 x_0 x_1 x_0^2 x_1 x_0 x_1 x_0.
\end{align*}
$$

We also use an explicit representation of the group $\Gamma_0$ by

$$
\begin{align*}
    x_0 &= A_0 + A_1 \frac{1}{t}, \\
    x_1 &= B_0 + B_1 \frac{1}{t},
\end{align*}
$$

where $A_0, A_1$ are certain $9 \times 9$ matrices over $\mathbb{F}_2$.

Rewriting this representation with finite band Toeplitz matrices and establishing certain periodicity patterns for higher commutators are at the heart of the proofs of Theorems 1 and 2.
We have $x_0 = A_0 + \frac{1}{y}A_1$ and $x_1 = B_0 + \frac{1}{y}B_1$ with $9 \times 9$ matrices $A_0, A_1, B_0, B_1 \in M(9, \mathbb{F}_2)$, and their inverses $x_0^{-1}, x_1^{-1}$ are of the same form. Therefore, an arbitrary group element $x \in G$ is of the form

$$x = C_0 + \sum_{j=1}^{k} \frac{1}{y^j}C_j,$$

which we identify with the (finite band) upper triangular infinite Toeplitz matrix.
What are expanders? Spectral properties of expanders New examples of expander families (joint work with N.Peyerimhoff) On SQ-universality of small cancellation

\[
x = \begin{pmatrix}
C_0 & C_1 & C_2 & \ldots & C_k & 0 & 0 & \ldots \\
0 & C_0 & C_1 & \ldots & C_{k-1} & C_k & 0 & \ddots \\
0 & 0 & C_0 & \ldots & C_{k-2} & C_{k-1} & C_k & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix},
\]  

where each $C_i$ is a matrix in $M(9, \mathbb{F}_2)$. One checks that multiplication of elements in $GL(9, \mathbb{F}_2[1/y])$ and of the corresponding infinite matrices is consistent.
A conjecture

MAGMA calculations support the following

**Conjecture (N.P., A.V.)**

Let $\lambda_n(\Gamma_0)$ be the lower exponent-2 central series of $\Gamma_0$, i.e.,

$$\lambda_{n+1} = [\lambda_n, \Gamma_0]\lambda_n^2.$$

Then we have the following 3-periodicity for the abelian quotients $\lambda_n / \lambda_{n+1}$, $n \geq 2$:

$$|\lambda_n / \lambda_{n+1}| = \begin{cases} 8, & \text{if } n \equiv 0, 1 \mod 3, \\ 4, & \text{if } n \equiv 2 \mod 3. \end{cases}$$
A conjecture

MAGMA calculations support the following

**Conjecture (N.P., A.V.)**

Let $\lambda_n(\Gamma_0)$ be the lower exponent-2 central series of $\Gamma_0$, i.e.,

$$\lambda_{n+1} = [\lambda_n, \Gamma_0] \lambda_n^2.$$

Then we have the following 3-periodicity for the abelian quotients $\lambda_n / \lambda_{n+1}$, $n \geq 2$:

$$|\lambda_n / \lambda_{n+1}| = \begin{cases} 8, & \text{if } n \equiv 0, 1 \mod 3, \\ 4, & \text{if } n \equiv 2 \mod 3. \end{cases}$$

This conjecture would imply that the tower of coverings of expander graphs in Theorem 2 would always have very low covering indices $4, 8, 4, 4, 8, 4, 4, 8, 4, 4, 8, 4, 4, \ldots$. 
A conjecture

MAGMA calculations support the following

**Conjecture (N.P., A.V.)**

Let \( \lambda_n(\Gamma_0) \) be the lower exponent-2 central series of \( \Gamma_0 \), i.e.,

\[
\lambda_{n+1} = [\lambda_n, \Gamma_0]\lambda_n^2.
\]

Then we have the following 3-periodicity for the abelian quotients \( \lambda_n / \lambda_{n+1}, n \geq 2 \):

\[
|\lambda_n / \lambda_{n+1}| = \begin{cases} 
8, & \text{if } n \equiv 0, 1 \text{ mod } 3, \\
4, & \text{if } n \equiv 2 \text{ mod } 3.
\end{cases}
\]
New girth and finite simple groups

Definition
Let $X = (V, E)$ be a finite $k$-regular graph with $n$ vertices. we say that $X$ is an $(n, k, c)$ expander if for any subset $A \subseteq V$

$$|\partial A| \geq c(1 - \frac{|A|}{N})|A|$$

where $\partial A = \{v \in V \mid d(v, A) = 1\}$. 
New girth and finite simple groups

**Definition**
Let \( X = (V, E) \) be a finite \( k \)-regular graph with \( n \) vertices. We say that \( X \) is an \((n, k, c)\) expander if for any subset \( A \subset V \)

\[
|\partial A| \geq c \left( 1 - \frac{|A|}{N} \right) |A|
\]

where \( \partial A = \{ v \in V \mid d(v, A) = 1 \} \).

**Theorem (Margulis)**
Let \( \Gamma \) be a finitely generated group that has property (T). Let \( \mathcal{L} \) be a family of finite index normal subgroups of \( \Gamma \) and let \( S = S^{-1} \) be a finite symmetric set of generators for \( \gamma \). Then the family \( \{ X(\Gamma / N, S) \mid N \in \mathcal{L} \} \) of Cayley graphs of the finite quotients of \( \Gamma \) with respect to the image of \( S \) is a family of \((n, k, c)\) expanders for \( n = |\Gamma / N|, k = |S| \) and some fixed \( c > 0 \).
New girth and finite simple groups

Definition
Let $\Gamma, L, S$ as above. Consider also the natural map $\phi_N : X(\Gamma, S) \to X(\Gamma/N, S)$. The new girth of a graph $X(\Gamma/N, S)$ is the length of the shortest circuit $\gamma$ in $X(\Gamma/N, S)$ so that $\gamma$ is not the image of a circuit in $X(\Gamma, S)$ under the map $\phi_N$. 
New girth and finite simple groups

**Definition**
Let $\Gamma, \mathcal{L}, S$ as above. Consider also the natural map $\phi_N : X(\Gamma, S) \to X(\Gamma/N, S)$. The *new girth* of a graph $X(\Gamma/N, S)$ is the length of the shortest circuit $\gamma$ in $X(\Gamma/N, S)$ so that $\gamma$ is not the image of a circuit in $X(\Gamma, S)$ under the map $\phi_N$.

**Theorem (R.Blok,C.Hoffman,AV)**
For any $n$ there exists an $\epsilon > 0$ and a symmetric set $S_{n,q}$ of generators for $SU_{2n}(q)$ so that $S_{n,q}$ has size five and the family of Cayley graphs $X(SU_{2n}(q), S_{n,q})$ for $q \geq n$ forms an $\epsilon$-expanding family of unbounded new girth.
Beauville surfaces

Definition

A Beauville surface $S$ is a complex algebraic surface, of the form $(C_1 \times C_2)/G$, where $C_1$ and $C_2$ are non-singular projective curves of genera $g(C_i) \geq 2$, and $G$ is a finite group acting freely on the product of curves by holomorphic transformations.
Beauville surfaces

**Definition**
A Beauville surface $S$ is a complex algebraic surface, of the form $(C_1 \times C_2)/G$, where $C_1$ and $C_2$ are non-singular projective curves of genera $g(C_i) \geq 2$, and $G$ is a finite group acting freely on the product of curves by holomorphic transformations.

**Definition**
A surface $S$ is said to be *isogenous to a product* if $S$ admits finite unramified covering isomorphic to a product of curves.
Beauville surfaces

**Definition**
Let $S$ be a surface isogenous to a product with minimal realisation $S \cong (C_1 \times C_2)/G$. We say that $S$ is a *mixed case* if the action of $G$ exchanges the two factors (and then $C_1$ and $C_2$ are isomorphic) and an *unmixed case* if $G$ acts via a diagonal action.
Beauville surfaces

Beauville’s original example had two curves $C_1 = C_2$, given by the Fermat curve $x^5 + y^5 + z^5 = 0$, and $G$ the group $(\mathbb{Z}/5\mathbb{Z})^2$ acting on $C_1 \times C_2$ by the rule

$$(a, b) \cdot ([x : y : z], [u : v : w]) = ([\xi^a x : \xi^b y : z], [\xi^{a+3b} u : \xi^{2a+4b} v : w]),$$

where $\xi = e^{\frac{2\pi i}{5}}$ and $a, b \in \mathbb{Z}/5\mathbb{Z}$. Then $S$ is a Beauville surface of unmixed type with $g(C_1) = g(C_2) = 6$. 
Beauville surfaces

Let $G$ be a finite group and $r$ an integer with $r \geq 2$. An $r$-tuple $T = [g_1, ..., g_r]$ of elements of $G$ is called a spherical system of generators, if $g_1, ..., g_r$ generate $G$ and we additionally have $g_1...g_r = 1$. For an $r$-tuple $T = [g_1, ..., g_r]$ of elements of $G$ and $g \in G$, we set

$$gTg^{-1} := [gg_1g^{-1}, ..., ggrg^{-1}].$$

If $A = [m_1, ..., m_r]$ is an $r$-tuple of natural numbers with $2 \leq m_1 \leq ... \leq m_r$, then the spherical system of generators $T = [g_1, ..., g_r]$ is said to be of type $A$, if there is a permutation $\tau \in \text{Sym}(r)$ such that we have

$$\text{ord}(g_1) = m_{\tau(1)}, \text{ord}(g_2) = m_{\tau(2)}, \ldots, \text{ord}(g_r) = m_{\tau(r)}.$$

(Here $\text{ord}(g)$ is the order of the element $g \in G$.)
Beauville surfaces

For a spherical system of generators \( T = [g_1, \ldots, g_r] \) of \( G \), we define

\[
\Sigma(T) = \Sigma([g_1, \ldots, g_r]) := \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \bigcup_{i=1}^{r} \{ g \cdot g_i^j \cdot g^{-1} \}
\]

(2)

to be the union of all conjugates of the cyclic subgroups generated by the elements \( g_1, \ldots, g_r \). A pair of spherical systems of generators \( (T_1, T_2) \) of \( G \) is called disjoint if

\[
\Sigma(T_1) \cap \Sigma(T_2) = \{1\}.
\]
Beauville surfaces

Next, we introduce unmixed and mixed ramification structures.

**Definition**
Let $A_1 = [m_{(1,1)}, \ldots, m_{(1,r)}]$ and $A_2 = [m_{(2,1)}, \ldots, m_{(2,s)}]$ be tuples of natural numbers with $2 \leq m_{(1,1)} \leq \ldots \leq m_{(1,r)}$ and $2 \leq m_{(2,1)} \leq \ldots \leq m_{(2,s)}$. An **unmixed ramification structure of type $(A_1, A_2)$** for $G$ is a disjoint pair $(T_1, T_2)$ of spherical systems of generators, such that $T_1$ has type $A_1$ and $T_2$ has type $A_2$. The disjointness of the pair $(T_1, T_2)$ in the definition of an unmixed ramification structure guarantees that $G$ acts freely on the product $C_{T_1} \times C_{T_2}$ of the associated algebraic curves.
Beauville surfaces

**Definition**

Let $A = [m_1, ..., m_r]$ be an $r$-tuple of natural numbers with $2 \leq m_1 \leq ... \leq m_r$. A *mixed ramification structure of type $A$ for $G$* is a pair $(H, T)$ where $H$ is a subgroup of index 2 in $G$ and $T = [g_1, ...g_r]$ is an $r$-tuple of elements of $G$ such that the following hold:
Beauville surfaces

Definition
Let $A = [m_1, ..., m_r]$ be an $r$-tuple of natural numbers with $2 \leq m_1 \leq ... \leq m_r$. A mixed ramification structure of type $A$ for $G$ is a pair $(H, T)$ where $H$ is a subgroup of index 2 in $G$ and $T = [g_1, ... g_r]$ is an $r$-tuple of elements of $G$ such that the following hold:

- $T$ is a spherical system of generators of $H$ of type $A$,
Beauville surfaces

Definition

Let $A = [m_1, \ldots, m_r]$ be an $r$-tuple of natural numbers with $2 \leq m_1 \leq \ldots \leq m_r$. A mixed ramification structure of type $A$ for $G$ is a pair $(H, T)$ where $H$ is a subgroup of index 2 in $G$ and $T = [g_1, \ldots, g_r]$ is an $r$-tuple of elements of $G$ such that the following hold:

- $T$ is a spherical system of generators of $H$ of type $A$,
- for every $g \in G \setminus H$, the spherical systems $T$ and $gTg^{-1}$ are disjoint,
Beauville surfaces

Definition
Let $A = [m_1, ..., m_r]$ be an $r$-tuple of natural numbers with $2 \leq m_1 \leq ... \leq m_r$. A mixed ramification structure of type $A$ for $G$ is a pair $(H, T)$ where $H$ is a subgroup of index 2 in $G$ and $T = [g_1, ..., g_r]$ is an $r$-tuple of elements of $G$ such that the following hold:

- $T$ is a spherical system of generators of $H$ of type $A$,
- for every $g \in G \setminus H$, the spherical systems $T$ and $gTg^{-1}$ are disjoint,
- for every $g \in G \setminus H$ we have $g^2 \notin \Sigma(T)$. 
Beauville surfaces

Definition

An *unmixed Beauville structure* is an unmixed ramification structure with two spherical systems \((T_1, T_2)\) of generators of length 3, i.e., \(r = 3\) and \(s = 3\).

A *mixed Beauville structure* is a mixed ramification structure with a spherical system \(T\) of generators of length 3, i.e., \(r = 3\).
Beauville surfaces

Let $G_1, G_2,...$ be the lower 2-central series for the group $\Gamma_0$ and $H_1, H_2,...$ be the lower 2-central series for $\Gamma$.

**Theorem (N.Barker, N.Boston, N.Peyerimhoff, AV)**

Let $3 \leq k \leq 64$. If $k$ is not a power of 2, then $\Gamma / H_k$ admits an unmixed Beauville structure.
Beauville surfaces

Let $G_1, G_2, \ldots$ be the lower 2-central series for the group $\Gamma_0$ and $H_1, H_2, \ldots$ be the lower 2-central series for $\Gamma$.

**Theorem (N.Barker, N.Boston, N.Peyerimhoff, AV)**

Let $3 \leq k \leq 64$. If $k$ is not a power of 2, then $\Gamma/H_k$ admits an unmixed Beauville structure.

**Theorem (N.Barker, N.Boston, N.Peyerimhoff, AV)**

Let $3 \leq k \leq 10$. If $k$ is not a power of 2, then $\Gamma_0/G_k$ admits a mixed Beauville structure.