McCool groups and stabilizers on the boundary of outer space

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Let $\mathcal{C}$ be a finite set of conjugacy classes in a free group F_n. Let $Out_{\mathcal{C}}(F_n)$ be the pointwise stabilizer of $\mathcal{C}$ in $Out(F_n)$. If for instance $\mathcal{C}$ is the class of $[a,b][c,d]$ in $F(a,b,c,d)$, then $Out_{\mathcal{C}}(F_n)$ is a mapping class group. We call $Out_{\mathcal{C}}(F_n)$ a McCool group because of:

Theorem 1 (McCool [1])**.** *$Out_{\mathcal{C}}(F_n)$ is finitely presented.*

McCool's proof used peak reduction. Using JSJ theory and outer space, we prove:

Theorem 2. *$Out_{\mathcal{C}}(F_n)$ is VFL: some finite index subgroup has a finite $K(\pi,1)$.*

Here is a sketch of the proof. It also applies if F_n is replaced by a torsion-free hyperbolic group.

One considers splittings of F_n (equivalently, graphs of groups decompositions, or actions on trees) which are relative to $\mathcal{C}$: every element of $\mathcal{C}$ must be contained in a vertex group (i.e. fix a point in the tree). There are two cases.

If F_n is freely indecomposable rel $\mathcal{C}$, one considers its cyclic JSJ decomposition rel $\mathcal{C}$. There is an $Out_{\mathcal{C}}(F_n)$-invariant JSJ tree T and one understands $Out_{\mathcal{C}}(F_n)$ through its action on T.

If F_n is not freely indecomposable rel $\mathcal{C}$, there is no $Out_{\mathcal{C}}(F_n)$-invariant tree and one has to consider outer space rel $\mathcal{C}$. This is the set of projective classes of actions of F_n on simplicial trees, with edge stabilizers trivial and vertex stabilizers freely indecomposable rel the elements of $\mathcal{C}$ which they contain. $Out_{\mathcal{C}}(F_n)$ acts "cocompactly" on this contractible space, and stabilizers are controlled by the previous case.

McCool groups come up when studying the action of $Out(F_n)$ on the boundary of (ordinary) outer space. A point on this boundary is a projective class $[T]$ of actions of F_n on $\mathbb{R}$-trees, and we distinguish between $Out_{[T]}(F_n)$ (the stabilizer of the projective tree) and $Out_T(F_n)$ (the stabilizer of the $\mathbb{R}$-tree). The quotient $Out_{[T]}(F_n)/Out_T(F_n)$ embeds into the multiplicative reals, and a result by M. Lustig implies that the image is trivial or cyclic. It it thus enough to study $Out_T(F_n)$.

Theorem 3. *$Out_T(F_n)$ has a finite index subgroup $Out^0_T(F_n)$ fitting in an exact sequence*

$$1 \to F_{n_1} \times \cdots \times F_{n_p} \to Out^0_T(F_n) \to M_1 \times \cdots \times M_q \to 1$$

where F_{n_i} is free and M_i is a McCool group.

In particular, $Out_{[T]}(F_n)$ and $Out_T(F_n)$ are VFL.

If for example T is the Bass-Serre tree of a cyclic amalgam $F *_C F'$ where the amalgam identifies $a \in F$ with $b \in F'$, the exact sequence is

$$1 \to \mathbb{Z} \to Out^0_T(F_n) \to Out_a(F) \times Out_b(F') \to 1$$

with the kernel generated by the Dehn twist (acting on F as conjugation by a and on F' as the identity).

If $[T]$ is fixed by an irreducible automorphism of F_n, then $Out_{[T]}(F_n)$ is virtually cyclic (Bestvina-Feighn-Handel).

To prove Theorem 3, one considers the preimage $Aut_T(F_n)$ of $Out_T(F_n)$ in $Aut(F_n)$. The action of F_n on T extends to an isometric action of $Aut_T(F_n)$, and we view an element H of $Aut_T(F_n)$ as an isometry of T.

If T is simplicial, the first step is to restrict to the finite index subgroup $PG(T) \subset Aut_T(F_n)$ consisting of elements acting trivially on the quotient graph T/F_n. The letters P and G stand for "Piecewise Group" because each element $H \in PG(T)$ *piecewise* agrees with an element of the *group* F_n: given an edge e of T, there exists $g \in F_n$ such that H and g agree on e.

In general, we define a subgroup $PG(T) \subset Aut_T(F_n)$ as follows: $H \in Aut_T(F_n)$ belongs to $PG(T)$ if and only if every arc in T may be subdivided into finitely many subarcs, and on each subarc H agrees with some element of F_n. Letting $\bar{P}G(T)$ be the image of $PG(T)$ in $Out_T(F_n)$, we show that $\bar{P}G(T)$ has finite index and admits a description as in Theorem 3.

References

[1] J. McCool, *Some finitely presented subgroups of the automorphism group of a free group*, Journal of Algebra **35** (1975), 205–213.