Geometry of infinite groups

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Contents

1 Preliminaries and notations

Nota bene: In order to ensure some coherence in the presentation, some definitions are not recalled in the text, but in a Dictionary at the end of the text.

Given X a set we denote by $\mathcal{P}(X)$ the power set of X. If two subsets A, B in X have the property that $A \cap B = \emptyset$ then we denote their union by $A \sqcup B$, and we call it *disjoint union*.

Let (X, dist) be a metric space.

For any $x \in X$ and $r > 0$ we denote by $B(x, r)$ the *open ball of center x and radius r*, i.e. the set $\{y \in X \; ; \; \mathrm{dist}(y, x) < r\}$.

We denote by $\overline{B}(x, r)$ the *closed ball of center x and radius r*, i.e. the set $\{y \in X \; ; \; \mathrm{dist}(y, x) \leq r\}$, and by $S(x, r)$ the *sphere of center x and radius r*, i.e. the set $\{y \in X \; ; \; \mathrm{dist}(y, x) = r\}$.

1.1 Length metric spaces

Throughout these notes by *path* we mean a continuous map $\mathfrak{p} : [a, b] \to X$. A path is said to *join* (or *connect*) two points x, y if $p(a) = x$, $p(b) = y$.

Given a *path* $\mathfrak{p}$ in X, one can define the *length* of the path p as follows. A partition $a = t_0 < t_1 < \quad < t_{n-1} < t_n = b$ of the interval $[a, b]$ defines a finite collection of points $\mathfrak{p}(t_0), \mathfrak{p}(t_1), , \mathfrak{p}(t_{n-1}), \mathfrak{p}(t_n)$ on the path $\mathfrak{p}$.

The *arc length of* $\mathfrak{p}$ is then defined to be

$$\mathrm{length}(\mathfrak{p}) = \sup_{a=t_0<t_1<\cdots<t_n=b} \sum_{i=0}^{n-1} \mathrm{dist}(\mathfrak{p}(t_i), \mathfrak{p}(t_{i+1}))$$

where the supremum is taken over all possible partitions of $[a, b]$ (n is an arbitrary integer).

By definition $\mathrm{length}(\mathfrak{p}) \geq \mathrm{dist}(\mathfrak{p}(a), \mathfrak{p}(b))$.

If the arc length of $\mathfrak{p}$ is finite then we say that $\mathfrak{p}$ is *rectifiable*, and we say that $\mathfrak{p}$ is *non-rectifiable* if the arc length is infinite.

Exercise 1.1. Consider a path in the Euclidean plane $\mathfrak{p} : [a, b] \to \mathbb{R}^2$, $\mathfrak{p}(t) = (x(t), y(t))$ which is moreover of class C^1. Prove that its arc length is

$$\mathrm{length}(\mathfrak{p}) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Exercise 1.2. Prove that the graph of the function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad 0 < x \leq 1\,, \\ 0 & \text{if} \quad x = 0\,, \end{cases}$$

is a non-rectifiable path joining $(0,0)$ and $(1, \sin 1)$.

Let (X, dist) be a metric space. We define a new metric dist_ℓ on X, known as the *induced intrinsic metric:* if two points x, y are joined by at least one rectifiable path then $\mathrm{dist}_\ell(x,y)$ is the infimum of the lengths of all rectifiable paths joining x to y; $\mathrm{dist}_\ell(x,y) = \infty$ otherwise.

We have that $\mathrm{dist} \leq \mathrm{dist}_\ell$.

A metric space (X, dist) such that $\mathrm{dist} = \mathrm{dist}_\ell$ is called a *length (or path) metric space.*

A priori a path realizing the infimum might not exist. If such a path exists then it is called a *geodesic.*

Exercise 1.3. Prove that a geodesic parameterized by arc-length, $\mathfrak{p} : [a,b] \to X$, is an isometric embedding.

Examples 1.4.

1. $\mathbb{R}^n$ with the Euclidean metric is a geodesic metric space.
2. $\mathbb{R}^n \setminus \{0\}$ with the Euclidean metric is a length metric space, but not a geodesic metric space.
3. The unit circle $\mathbb{S}^1$ with the metric inherited from the Euclidean metric of $\mathbb{R}^2$ (the chordal metric) is not a length metric space. The induced intrinsic metric on $\mathbb{S}^1$ measures distances as angles in radians.
4. Every Riemannian manifold can be turned into a path metric space by defining the distance of two points as the infimum of the lengths of the C^1 curves connecting the two points.

Theorem 1.5 (Hopf-Rinow theorem [Gro07])**.** *If a length metric space is complete and locally compact then it is geodesic and proper.*

1.2 Graphs and simplicial trees

Recall that a *(simple) graph* consists of a pair of data:

- a set V called *set of vertices* of the graph;
- a set E of unordered pairs of distinct vertices $\{u, v\}$, called *edges* of the graph.

Note that, compared to a more general definition (of the sometimes called *multigraphs*), we excluded the existence of self-loops (i.e. edges $\{u, u\}$) and of multiple edges (i.e. repetitions of the same pair in E). The vertices u, v belonging to an edge $\{u, v\}$ are called the *endpoints* of the edge.

Two u, v vertices such that $\{u, v\}$ is an edge are called *adjacent.*

The *valency (or degree) of a vertex* is the number of vertices adjacent to it. A *regular graph* is a graph where each vertex has the same valency. A regular graph with vertices of valency k is called a *k–regular graph* or *regular graph of degree k.*

A *chain* or *path* is a sequence of edges $\{u_1, v_1\}, \ldots, \{u_n, v_n\}$ such that $\{u_i, v_i\} \cap \{u_{i+1}, v_{i+1}\} \neq \emptyset$ for every $i \in \{1, ..., n-1\}$. If moreover $\{u_n, v_n\} \cap \{u_1, v_1\} \neq \emptyset$ then the path is called a *collapsed cycle.*

Given a finite set of pairwise distinct vertices, $v_1, v_2, ..., v_n$, the path $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_n, v_1\}$ is called a *cycle*.

A graph is *connected* if for every two distinct vertices x, y there exists a path $\{u_1, v_1\}, \ldots, \{u_n, v_n\}$ with $u_1 = x$ and $v_n = y$.

A *simplicial tree* is a connected graph without cycles.

A *directed graph* (or a *digraph*) is defined by:

- a set V called *set of vertices* of the graph;
- a subset E of $V \times V \setminus \{(v, v)\,;\, v \in V\}$, called *set of directed edges* of the graph.

A directed graph is called *symmetric* if for every $(u, v) \in E$ the ordered pair (v, u) is also in E. Thus E is stable with respect to the involution $i(x, y) = (y, x)$.

A symmetric directed graph is equivalent to a (non-oriented) graph with the same set of vertices, via the replacement of each pair of oriented edges $(u, v), (v, u)$ by the non-oriented edge $\{u, v\}$. The non-oriented graph thus obtained is called the *underlying non-oriented graph* of the given directed graph.

1.3 The real hyperbolic plane

A good reference for this section and for hyperbolic spaces in general is [And05].

The real hyperbolic space appeared due to the following question:

Question 1.6. *Does Euclid's fifth postulate, also called the parallel postulate (an equivalent version of which is: through any point not on a given line can be drawn one and only one line parallel to the given line) follow from the other postulates ?*

N.I. Lobachevski, J. Bolyai and C.F. Gauss independently developed a theory of non-Euclidean geometry, where Euclid's fifth postulate was replaced by 'through any point not on a given line can be drawn *at least one* line not intersecting the given line'. The independence of the parallel postulate from Euclid's other postulates was proved by E. Beltrami in 1868, via the construction of a model of a geometry in which the other postulates are satisfied, but not the fifth. Here we present another model, due to Poincaré.

The *Poincaré half-plane* is the set $\mathbb{H}^2 = \{z \in \mathbb{C}\,;\, \mathrm{Im}\, z > 0\}$ endowed with the metric:

$$\mathrm{dist}\,(z, w) = \mathrm{arccosh}\left(1 + \frac{|z - w|^2}{2\mathrm{Im}\, z \mathrm{Im}\, w}\right). \tag{1}$$

Equivalently,

$$\mathrm{dist}\,(z, w) = 2\mathrm{arcsinh}\left(\frac{|z - w|^2}{2\mathrm{Im}\, z \mathrm{Im}\, w}\right). \tag{2}$$

Exercise 1.7. Deduce from (1) that

$$\ln\left(1 + \frac{|z - w|^2}{2\mathrm{Im}\, z \mathrm{Im}\, w}\right) \leq \mathrm{dist}\,(z, w) \leq \ln\left(1 + \frac{|z - w|^2}{2\mathrm{Im}\, z \mathrm{Im}\, w}\right) + \ln 2\,.$$

The group $SL(2, \mathbb{R})$ acts on this set by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}\,. \tag{3}$$

Exercise 1.8. Prove that (3) defines an action of $SL(2, \mathbb{R})$ on $\mathbb{H}^2$.

Note that the matrices A and $-A$ define the same transformation of $\mathbb{H}^2$.

1.3.A The action of $SL(2,\mathbb{R})$ on $\mathbb{H}^2$ is by isometries.

Exercise 1.9. Prove that any matrix in $SL(2,\mathbb{R})$ is either of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ or it can be written as a product $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Hint: If a matrix is not of the first type then it is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \neq 0$. Use this information and multiplications to the left and to the right with matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ to create zeroes in the matrix.

Exercise 1.10.
1. Prove that a matrix $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ acts as an isometry on $\mathbb{H}^2$.
2. Prove that the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts as an isometry on $\mathbb{H}^2$. Deduce that $SL(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by isometries.

1.3.B The action of $SL(2,\mathbb{R})$ on $\mathbb{H}^2$ is transitive.

The action of $SL(2,\mathbb{R})$ on $\mathbb{H}^2$ is *transitive* if and only if for every $z, w \in \mathbb{H}^2$ there exists $A \in SL(2,\mathbb{R})$ such that $Az = w$.

This follows easily from the fact that for every $x + iy \in \mathbb{H}^2$

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i = x + iy$$

In fact the above shows that the subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \; ; \; a \in (0,\infty), b \in \mathbb{R} \right\} \tag{4}$$

acts transitively on $\mathbb{H}^2$.

We shall now apply the philosophy of Klein's Erlangen programme and rediscover some of the essential features of hyperbolic geometry via the action of its group of isometries.

1.3.C The hyperbolic circles are Euclidean circles.

A hyperbolic circle is a set of the form

$$C_h(w,\delta) = \{z \in \mathbb{H}^2 \; ; \; \mathrm{dist}(z,w) = \delta\}$$

while an Euclidean circle is a set of the form

$$C_e(w,d) = \{z \in \mathbb{H}^2 \; ; \; |z - w| = d\} \, .$$

Exercise 1.11. Prove that the hyperbolic circle of centre i and radius δ is the same as the Euclidean circle of centre $i\cosh\delta$ and radius $\sinh\delta$.
Hint. It follows immediately from the distance formula (1).

Exercise 1.12. Prove that the transformation defined by the formula (3) for a matrix $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $a \in (0,\infty), b \in \mathbb{R}$, transforms an Euclidean circle into another Euclidean circle.

The two exercises above and the fact that the subgroup P defined in formula (4) acts transitively on $\mathbb{H}^2$ imply that any hyperbolic circle is an Euclidean circle.

1.3.D The group $SL(2,\mathbb{R})$ acts transitively on pairs of equidistant points.

Exercise 1.13.
- Prove that the stabilizer of the point i in $SL(2,\mathbb{R})$ is $SO(2)$.
- Prove that $SO(2)$ acts transitively on $C_h(i,r)$ for any $r > 0$.

Let x, y be two points at hyperbolic distance δ. Note that i and ie^δ are two points with the same property. Since $SL(2,\mathbb{R})$ acts transitively on $\mathbb{H}^2$ there exists A such that $Ax = i$. Then $Ay \in C_h(i,\delta)$. Exercise 1.13 implies that there exists $B \in SO(2)$ such that $BAy = ie^\delta$, while $BAx = Bi = i$

1.3.E The geodesics are either segments of vertical lines or sub-arcs in circles with centre on the line $\operatorname{Im} z = 0$.

We shall apply the transitivity on pairs of equidistant points, and start by describing geodesics joining the point i to a point ie^δ.

Exercise 1.14. Using the distance formula (1) prove that

- $\mathfrak{g} : [0,\delta] \to \mathbb{H}^2, g(t) = ie^t$ is a geodesic;
- for every point z outside the above geodesic $\operatorname{dist}(i,z) + \operatorname{dist}(z, ie^\delta) > \delta$.

We can thus conclude that for any $\delta \in \mathbb{R}$ there exists a unique geodesic joining i and ie^δ, and that is the vertical segment.

Exercise 1.15. By eventually using the result in Exercise 1.9 prove that every $A \in SL(2,\mathbb{R})$ transforms a vertical half-line either into another vertical half-line or into the intersection of $\mathbb{H}^2$ with a circle with centre on the line $\operatorname{Im} z = 0$.

An immediate consequence of Exercises 1.14 and 1.15 and of the fact that $SL(2,\mathbb{R})$ acts transitively on pairs of equidistant points is the following:

Proposition 1.16. *Given any two points $z, w \in \mathbb{H}^2$ there exists a unique geodesic joining them, and that geodesic is either a segment of a vertical line or a sub-arc in a circle with centre on the line* $\operatorname{Im} z = 0$.

1.3.F $\mathbb{H}^2$ is a symmetric space.

A *symmetric space* is a complete simply connected Riemannian manifold X such that for every point p there exists a global isometry of X which is a geodesic symmetry σ_p with respect to p, that is for every geodesic $\mathfrak{g}$ through p, $\sigma_p(\mathfrak{g}(t)) = \mathfrak{g}(-t)$. Details on the notion can be found in [Hel01].

Exercise 1.17. 1. Prove that the transformation σ_i defined by $\pm S_i$, where $S_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ fixes i and restricted to the vertical line through i it is a symmetry.

2. Prove that for any $O \in SO(2)$, $O^T S_i O = S_i$.

3. Deduce from the above and Exercise 1.13 that σ_i is a geodesic symmetry with respect to i.

From Exercise 1.17 and the transitive action of $SL(2,\mathbb{R})$ we immediately deduce that for every point $p \in \mathbb{H}^2$ there exists a global isometry of $\mathbb{H}^2$ which is a geodesic symmetry.

Remark 1.18. In fact $\mathbb{H}^2$ is a symmetric space *of non-positive sectional curvature and of rank one.*

We shall explain the latter in the next section.

Definition 1.19. The *rank of a symmetric space* (of non-positive sectional curvature) is the maximal $n \in \mathbb{N}$ such that the n–dimensional Euclidean space has an isometric copy in the symmetric space which is *totally geodesic* (i.e. it contains, together with any pair of points, all the geodesics connecting them).

1.3.G $\mathbb{H}^2$ behaves like a bounded perturbation of a real tree.

Theorem 1.20. *(see for instance [GdlH90], Section 2, Corollary 22) For any geodesic triangle in $\mathbb{H}^2$, each edge is contained in the tubular neighbourhood of radius* $\ln 3$ *of the union of the other two edges.*

We say that geodesic triangles in $\mathbb{H}^2$ are δ*–thin*, with $\delta = \ln 3$. Later on, we shall discuss the proof of Theorem 1.20 in more detail.

Remark 1.21. Similarly, for any geodesic k–gon, each edge is contained in the δ_k–tubular neighbourhood of the union of the other edges.

Exercise 1.22. Deduce from the above that $\mathbb{H}^2$ has rank one.

1.4 From non-discrete to discrete: quasi-isometries

We define an important equivalence relation between metric spaces: the quasi-isometry. We keep in mind that $\mathbb{R}^n$ with the Euclidean metric should be made equivalent to $\mathbb{Z}^n$ with the length metric on the grid with $\mathbb{Z}^n$ as set of vertices.

The quasi-isometry equivalence relation has two equivalent definitions: one which is easy to visualize and one which makes it easier to understand why it is an equivalence relation. We begin by the first definition, continue by the second and prove their equivalence.

A map $f : X \to Y$ of a metric space (X, dist_X) into a metric space (Y, dist_Y) is called *L–Lipschitz*, or simply *Lipschitz* if

$$\mathrm{dist}_Y(f(x_1), f(x_2)) \leq L\mathrm{dist}_X(x_1, x_2)\,.$$

The map is called *L–bi-Lipschitz*, or simply *bi-Lipschitz* if

$$\frac{1}{L}\mathrm{dist}_X(x_1,x_2) \le \mathrm{dist}_Y(f(x_1),f(x_2)) \le L\mathrm{dist}_X(x_1,x_2)\,.$$

Note that f is necessarily one-to-one. If moreover f is onto, the metric spaces (X, dist_X) and (Y, dist_Y) are called *bi-Lipschitz equivalent*.

If dist_1 and dist_2 are two distances on the same metric space X such that the identity map $\mathrm{id} : (X, \mathrm{dist}_1) \to (X, \mathrm{dist}_2)$ is bi-Lipschitz, then we say that dist_1 and dist_2 are *bi-Lipschitz equivalent*.

Examples 1.23.

1. The metrics on $\mathbb{R}^n$ induced by the Euclidean norm $\|\cdot\|_e$ and by the norm $\|(x_1,...,x_n)\|_1 = |x_1| + \cdots + |x_n|$ are bi-Lipschitz equivalent. The same holds for $\|\cdot\|_1$ replaced by $\|\cdot\|_{max}$, where $\|(x_1,...,x_n)\|_{max} = \max\{|x_1|,...,|x_n|\}$.
2. Two distances on the same finite dimensional real vector space defined by two scalar products are bi-Lipschitz equivalent.
3. Two left-invariant Riemannian metrics on a connected real Lie group are bi-Lipschitz equivalent.

Let (X, dist) be a metric space.

Definition 1.24. An *ε–separated set A* in X is a set such that

$$\mathrm{dist}(a_1,a_2) \ge \varepsilon\,,\ \forall a_1,a_2 \in A\,.$$

A *δ–net* in X is a subset B in X such that X is contained in the δ–tubular neighbourhood of B, i.e. for every $x \in X$ there exists $b \in B$ such that $\mathrm{dist}(x,b) < \delta$.

Definition 1.25. We call an ε–separated δ–net in X a *separated net*.

Remark 1.26. A maximal δ–separated set in X is a δ–net in X.

Proof. Let N be a maximal δ–separated set in X. For every $x \in X \setminus N$, the set $N \cup \{x\}$ is no longer δ–separated, by maximality of N. Hence there exists $y \in N$ such that $\mathrm{dist}(x,y) < \delta$. □

By Zorn's lemma a maximal δ–separated set always exists. Thus every metric space contains a δ–separated net, for any $\delta > 0$.

Exercise 1.27. Prove that if (X, dist) is compact then every separated net is finite, hence every separated set is finite.

Definition 1.28. Two metric spaces (X, dist_X) and (Y, dist_Y) are *quasi-isometric* if and only if there exist $A \subset X$ and $B \subset Y$ separated nets such that (A, dist_X) and (B, dist_Y) are bi-Lipschitz equivalent.

Examples 1.29.

1. A metric space of finite diameter is quasi-isometric to a point.
2. The space $\mathbb{R}^n$ endowed with a norm $\|\cdot\| = \|\cdot\|_e$ or $\|\cdot\|_1$ or $\|\cdot\|_{max}$, is quasi-isometric to $\mathbb{Z}^n$ with the induced metric.

In general quasi-isometry is defined to formalize the relationship between some discrete metric spaces (most of the time groups) and some "non-discrete" (or continuous) metric spaces like for instance Riemannian manifolds etc. A particular instance of this is the relationship between hyperbolic spaces and some hyperbolic groups, which we will discuss further on.

When trying to prove that the quasi-isometry relation is an equivalence relation, reflexivity and symmetry are straightforward, but when attempting to prove transitivity, the following question naturally arises:

Question 1.30 ([Gro93], p. 23). *Can a space contain two separated nets that are not bi-Lipschitz equivalent ?*

Theorem 1.31 ([BK98]). *There exists a separated net N in $\mathbb{R}^2$ which is not bi-Lipschitz equivalent to $\mathbb{Z}^2$.*

Open question 1.32 ([BK02]). *When placing a point in the barycentre of each tile of a Penrose tiling, is the resulting separated net bi-Lipschitz equivalent to $\mathbb{Z}^2$?*

A more general version of it: embed $\mathbb{R}^2$ into $\mathbb{R}^n$ as a plane P with irrational slope and take B bounded subset with non-empty interior. Consider all $z \in \mathbb{Z}^n$ such that $z + B$ intersects P. The projections of all such z on P compose a separated net. Is such a net bi-Lipschitz equivalent to $\mathbb{Z}^2$?

Fortunately there is a second equivalent way of defining the fact that two metric spaces are quasi-isometric, which is as follows.

An (L, C)-*quasi-isometric embedding* (or simply a *quasi-isometric embedding*) of a metric space (X, dist_X) into a metric space (Y, dist_Y) is a map $\mathfrak{q} : X \to Y$ such that for every $x_1, x_2 \in X$,

$$\frac{1}{L}\mathrm{dist}_X(x_1, x_2) - C \leq \mathrm{dist}_Y(\mathfrak{q}(x_1), \mathfrak{q}(x_2)) \leq L\mathrm{dist}_X(x_1, x_2) + C\,, \tag{5}$$

for some constants $L \geq 1$ and $C \geq 0$.

If X is a finite interval $[a, b]$ then $\mathfrak{q}$ is called *a quasi-geodesic (segment).* If $a = -\infty$ or $b = +\infty$ then $\mathfrak{q}$ is called *quasi-geodesic ray.* If both $a = -\infty$ and $b = +\infty$ then $\mathfrak{q}$ is called *quasi-geodesic line.* The same names are used for the image of $\mathfrak{q}$.

If moreover Y is contained in the C–tubular neighborhood of $\mathfrak{q}(X)$ then $\mathfrak{q}$ is called an (L, C)–*quasi-isometry* (or simply *a quasi-isometry*).

Proposition 1.33. *Two metric spaces (X, dist_X) and (Y, dist_Y) are quasi-isometric if and only if there exists a quasi-isometry $\mathfrak{q} : X \to Y$.*

Proof. Assume there exists an (L, C)–quasi-isometry $\mathfrak{q} : X \to Y$. Let $\delta = L(C + 1)$ and let A be a δ–separated ε–net. Then $B = \mathfrak{q}(A)$ is a 1–separated $(L\varepsilon + 2C)$–net. Moreover for any $a, a' \in A$, $\mathrm{dist}_Y(\mathfrak{q}(a), \mathfrak{q}(a')) \leq L\mathrm{dist}_X(a, a') + C \leq \left(L + \frac{C}{\delta}\right) \mathrm{dist}_X(a, a')$ and $\mathrm{dist}_Y(\mathfrak{q}(a), \mathfrak{q}(a')) \geq \frac{1}{L}\mathrm{dist}_X(a, a') - C \geq \left(\frac{1}{L} - \frac{C}{\delta}\right) \mathrm{dist}_X(a, a') = \frac{1}{L(C+1)}\mathrm{dist}_X(a, a')$. It follows that $\mathfrak{q}$ restricted to A and with target B is bi-Lipschitz.

Conversely assume that $A \subset X$ and $B \subset Y$ are two ε–separated δ–nets, and that there exists a bi-Lipschitz map $f : A \to B$ which is onto. We define a map $\mathfrak{q} : X \to Y$ as follows: for every $x \in X$ we choose one $a_x \in A$ at distance at most δ from x and define $\mathfrak{q}(x) = f(a_x)$.

N.B. The axiom of choice makes here its first important appearance, if we do not count the episodic appearance of Zorn's Lemma, which is equivalent to the axiom of choice. Details on this axiom will be provided later on. Still, when X is proper there are finitely many possibilities for

a_x, so the axiom of choice need not be assumed, in the finite case it follows from the Zermelo–Fraenkel axioms.

Since $\mathfrak{q}(X) = \mathfrak{q}(A) = B$ it follows that Y is contained in the ε–tubular neighbourhood of $\mathfrak{q}(X)$. For every $x, y \in X$, $\mathrm{dist}_Y(\mathfrak{q}(x), \mathfrak{q}(y)) = \mathrm{dist}_Y(f(a_x), f(a_y)) \leq L\mathrm{dist}_X(a_x, a_y) \leq L(\mathrm{dist}_X(x, y) + 2\varepsilon)$. Also $\mathrm{dist}_Y(\mathfrak{q}(x), \mathfrak{q}(y)) = \mathrm{dist}_Y(f(a_x), f(a_y)) \geq \frac{1}{L}\mathrm{dist}_X(a_x, a_y) \geq \frac{1}{L}(\mathrm{dist}_X(x, y) - 2\varepsilon)$. □

With this second definition of quasi-isometry reflexivity and transitivity are immediate, symmetry amounts to the following property:

Proposition 1.34. *Consider two metric spaces (X, dist_X) and (Y, dist_Y), and a quasi-isometry $\mathfrak{q} : X \to Y$.*

There exists a quasi-isometry $\bar{\mathfrak{q}} : Y \to X$ such that $\bar{\mathfrak{q}} \circ \mathfrak{q}$ and $\mathfrak{q} \circ \bar{\mathfrak{q}}$ are at uniformly bounded distance from the respective identity maps. We call $\bar{\mathfrak{q}}$ quasi-converse *of $\mathfrak{q}$.*

Exercise 1.35. Prove Proposition 1.34, using Proposition 1.33 or any other argument.

2 Free groups, finitely generated groups, Cayley graphs

2.1 Definition of a free group

In the sequel we use the following

Convention: In a group G we denote its neutral element either by id or by 1.

Let X be a set, called *alphabet*. Its elements are called *letters* or *symbols*. We define the set of *inverse letters* (or *inverse symbols*) $X^{-1} = \{a^{-1} \mid a \in X\}$. A *word* in $X \cup X^{-1}$ is a finite (possibly empty) sequence of elements in $X \cup X^{-1}$. The *length* of the word is its length as a sequence. The length of the empty word is 0. A word is *reduced* if it contains no pair of consecutive letters of the form aa^{-1} or $a^{-1}a$. The *reduction* of a word in $X \cup X^{-1}$ is the deletion of all pairs of consecutive letters of the form aa^{-1} or $a^{-1}a$.

We define an equivalence relation between words in $X \cup X^{-1}$ by $w \sim w'$ if w can be obtained from w' by a finite sequence of deletions of pairs of consecutive letters of the form aa^{-1} or $a^{-1}a$, or insertions of such pairs.

Proposition 2.1. *Any word in $X \cup X^{-1}$ is equivalent to a unique reduced word.*

Proof. Existence. We prove the statement by induction on the length of a word. For words of length 0 and 1 the statement is clearly true. Assume that it is true for words of length n and consider a word of length $n+1$, $w = a_1....a_na_{n+1}$, where $a_i \in X \cup X^{-1}$. According to the induction hypothesis there exists a reduced word $u = b_1...b_k$ with $b_j \in X \cup X^{-1}$ such that $a_2....a_{n+1} \sim u$. Then $w \sim a_1u$. If $a_1 \neq b_1^{-1}$ then a_1u is reduced. If $a_1 = b_1^{-1}$ then $a_1u \sim b_2...b_k$ and the latter word is reduced.

Uniqueness. Let $F(X)$ be the set of reduced words in $X \cup X^{-1}$. For every $a \in X \cup X^{-1}$ we define a map $L_a : F(X) \to F(X)$ by

$$L_a(b_1...b_k) = \begin{cases} ab_1...b_k & \text{if} \quad a \neq b_1^{-1}, \\ b_2...b_k & \text{if} \quad a = b_1^{-1}. \end{cases}$$

For every word $w = a_1....a_n$ define $L_w = L_{a_1} \circ \cdots \circ L_{a_n}$. For the empty word e define $L_e = \mathrm{id}$. It is easy to check that $L_a \circ L_{a^{-1}} = \mathrm{id}$ for every $a \in X \cup X^{-1}$, and to deduce from it that $v \sim w$ implies $L_v = L_w$.

We prove by induction on the length that if w is reduced then $w = L_w(e)$. The statement clearly holds for w of length 0 and 1. Assume that it is true for reduced words of length n and let w be a reduced word of length $n+1$. Then $w = au$, where $a \in X \cup X^{-1}$ and u is a reduced word that does not begin with a^{-1}, i.e. such that $L_a(u) = au$. Then $L_w(e) = L_a \circ L_u(e) = L_a(u) = au = w$.

In order to prove uniqueness it suffices to prove that if $v \sim w$ and v, w are reduced then $v = w$. Since $v \sim w$ it follows that $L_v = L_w$, hence $L_v(e) = L_w(e)$, that is $v = w$. □

Definition 2.2. The *free group over* X, $F(X)$, is the set of reduced words in $X \cup X^{-1}$, endowed with the product defined by: $w * w'$ is the unique reduced word equivalent to the word ww'. The unit is the empty word.

The cardinality of X is called the *rank* of the free group $F(X)$.

Exercise 2.3. Prove that $F(X)$ with the product defined in Definition 2.2 is a group.

Remark 2.4. A free group of rank at least two is not Abelian. Thus *free non-Abelian* means free of rank at least two.

2.2 Ping-pong lemma. Examples of free groups.

Lemma 2.5 (the Ping-pong (or Table-Tennis) lemma)**.** *Let X be a space, and let $g : X \to X$ and $h : Y \to Y$ be two maps one-to-one and onto. If A, B are two non-empty subsets of X, such that $A \not\subset B$ and if*

$$g^n(A) \subset B \text{ for every } n \in \mathbb{Z} \setminus \{0\},$$

$$h^m(B) \subset A \text{ for every } m \in \mathbb{Z} \setminus \{0\},$$

then g, h generate a free subgroup of rank 2 in the group of one-to-one onto maps $X \to X$ with the binary relation $\circ$.

Proof. Step 1. Let w be a non-empty word in $\{g, g^{-1}, h, h^{-1}\}$. We want to prove that $w \neq 1$. We begin by noting that it is enough to prove this when

$$w = g^{n_1} h^{n_1} g^{n_2} h^{n_2} \dots g^{n_k}, \text{ with } n_j \in \mathbb{Z} \setminus \{0\} \, \forall j \in \{1, ..., k\}. \tag{6}$$

Indeed:

- If $w = h^{n_1} g^{n_2} h^{n_2} \dots g^{n_k} h^{n_k}$ then gwg^{-1} is as in (6), and $gwg^{-1} \neq \mathrm{id} \Rightarrow w \neq \mathrm{id}$.
- If $w = g^{n_1} h^{n_1} g^{n_2} h^{n_2} \dots g^{n_k} h^{n_k}$ then for any $m \neq -n_1$, $g^m w g^{-m}$ is as in (6).
- If $w = h^{n_1} g^{n_2} h^{n_2} \dots g^{n_k}$ then for any $m \neq n_k$, $g^m w g^{-m} \neq \mathrm{id}$ is as in (6).

Step 2. If w is as in (6) then

$$w(A) \subset g^{n_1} h^{n_1} g^{n_2} h^{n_2} \dots g^{n_{k-1}} h^{n_{k-1}}(B) \subset g^{n_1} h^{n_1} g^{n_2} h^{n_2} \dots g^{n_{k-1}}(A) \subset \dots \subset g^{n_1}(A) \subset B.$$

If $w = \mathrm{id}$ then it would follow that $A \subset B$, a contradiction. □

Example 2.6. For any integer $k \geq 2$ the matrices

$$g = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

generate a free subgroup of $SL(2, \mathbb{Z})$.

1st proof. The group $SL(2,\mathbb{Z})$ acts on $\mathbb{H}^2$. The matrix g acts as a horizontal translation $z \mapsto z+k$, while

$$h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore h acts as represented in Figure 1. We apply Lemma 2.5 to g, h and the subsets A and B represented below, i.e. A is the strip $\{z \in \mathbb{H}^2 \ ; \ -\frac{k}{2} < \operatorname{Re} z < \frac{k}{2}\}$ and B is the complementary of its closure, that is $B = \{z \in \mathbb{H}^2 \ ; \ \operatorname{Re} z < -\frac{k}{2} \text{ or } \operatorname{Re} z > \frac{k}{2}\}$.

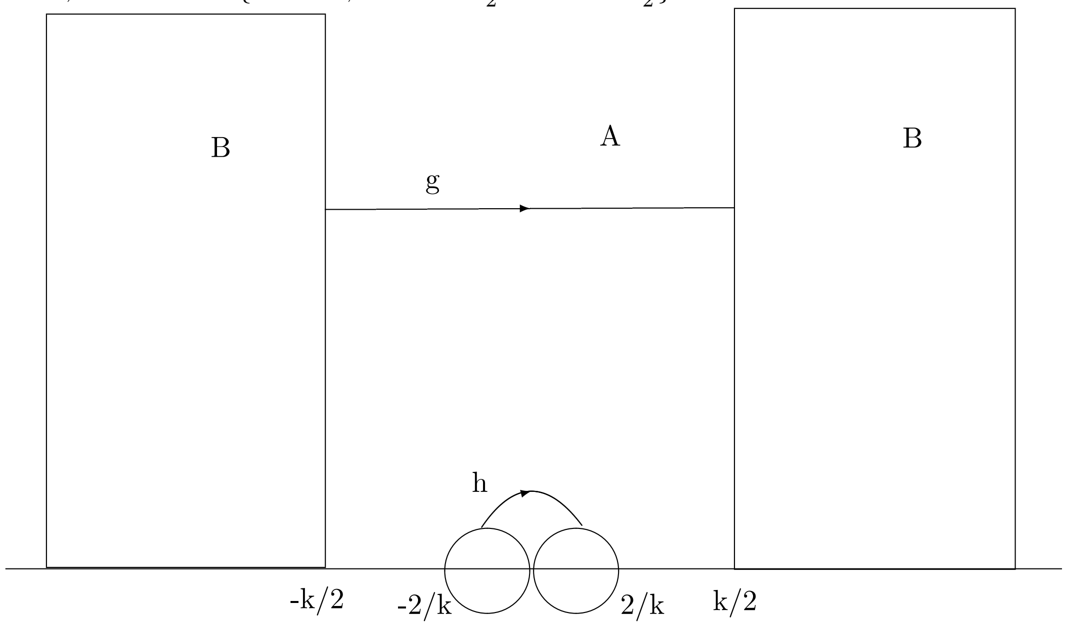

Figure 1: Example of ping-pong.

2nd proof. The group $SL(2,\mathbb{Z})$ also acts linearly on $\mathbb{R}^2$, and we can apply Lemma 2.5 to g, h and the subsets of $\mathbb{R}^2$

$$A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ ; \ |x| < |y| \right\} \text{ and } B = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ ; \ |x| > |y| \right\}.$$

Remark 2.7. The statement no longer holds for $k = 1$. Indeed in that case we have

$$g^{-1}hg^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus $(g^{-1}hg^{-1})^2 = \mathrm{Id}_2$, hence the group generated by g, h is not free.

Example 2.8 ([Wag85], Theorem 2.1). Let $\theta = \arccos\frac{1}{3}$. The following matrices in $SO(3)$ generate a free subgroup of rank 2:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Proof. It suffices to prove that any reduced word in $\{A^{\pm 1}, B^{\pm 1}\}$ is not I_3. Up to conjugation we may assume that the last letter in w is A. We prove by induction on the length that if w is a word with the last letter A then the first column of w is of the form $\frac{1}{3^m}\left(a, b\sqrt{2}, c\right)$, where m is the length of w, a, b, c are integers and b is not divisible by 3 (in particular not 0).

When $|w| = 1$ we have $w = A$, whose first column is $\frac{1}{3}\left(1, 2\sqrt{2}, 0\right)$. Assume that the statement holds for $|w| \leq n$ and let w be a reduced word of length $n+1$. Then $w = lu$, where u is a reduced word of length n and $l = A^{\pm 1}$ or $B^{\pm 1}$. By the induction hypothesis the first column of w is $we_1 = A^{\pm 1}X$ or $B^{\pm 1}X$, where X is a vector of the form $\frac{1}{3^m}\left(a, b\sqrt{2}, c\right)^T$, with m, a, b, c integers. It follows that $we_1 = \frac{1}{3^{m+1}}\left(a', b'\sqrt{2}, c'\right)^T$, where $a' = a \mp 4b$, $b' = b \pm 2a$, $c' = 3c$ if $l = A^{\pm 1}$, and $a' = 3a$, $b' = b \mp 2c$, $c' = c \pm 4b$ if $l = B^{\pm 1}$.

We now prove that b is not divisible by 3. If u begins also by l, that is $u = lv$, and $ve_1 = \frac{1}{3^{m-1}}\left(a_1, b_1\sqrt{2}, c_1\right)^T$ then direct computation shows that $b' = 2b - 9b_1$, hence b' is not divisible by 3 because b is not divisible by 3.

If u does not begin by l then we have that $w = A^{\pm 1}B^{\pm 1}v$ or $w = B^{\pm 1}A^{\pm 1}v$. In the first case $b' = b \pm 2a$ and a is divisible by 3, in the second base $b' = b \mp 2c$ and c is divisible by 3. In both cases since b is not divisible by 3 it follows that b' is not divisible by 3. □

2.3 The rank of a free group determines the group. Subgroups

Proposition 2.9 (Universal property of free groups)**.** *A map $\varphi : X \to G$ from the set X to a group G can be extended in a unique way to a homomorphism $\Phi : F(X) \to G$.*

Proof. Existence. The map φ can be extended to a map on $X \cup X^{-1}$ (which we denote also φ) by $\varphi(a^{-1}) = \varphi(a)^{-1}$.

For every reduced word $w = a_1...a_n$ in $F(X)$ define $\Phi(a_1...a_n) = \varphi(a_1)\dots\varphi(a_n)$. Define also $\Phi(e) = 1$, the identity element of G. It is easy to check that Φ is a homomorphism.

Uniqueness. Let $\Psi : F(X) \to G$ be a homomorphism such that $\Psi(x) = \varphi(x)$ for every $x \in X$. Then for every reduced word $w = a_1...a_n$ in $F(X)$, $\Psi(w) = \Psi(a_1)\dots\Psi(a_n) = \varphi(a_1)\dots\varphi(a_n) = \Phi(w)$. □

Corollary 2.10. *Every group is a quotient of a free group.*

Proof. Apply Proposition 2.9 to the group G and the set $X = G$. □

Proposition 2.11. *Two free groups $F(X)$ and $F(Y)$ are isomorphic if and only if X and Y have the same cardinality.*

Proof. A one-to-one onto map $\varphi : X \to Y$ extends to an isomorphism $\Phi : F(X) \to F(Y)$ by Proposition 2.9. Therefore two free groups $F(X)$ and $F(Y)$ are isomorphic if X and Y have the same cardinality.

Conversely, assume $F(X)$ and $F(Y)$ are isomorphic. Let $\Phi : F(X) \to F(Y)$ be an isomorphism. Take $N(X) \leq F(X)$ the normal subgroup generated by $\{g^2\,;\, g \in F(X)\}$. Then $\Phi(N(X)) = N(Y)$ is the normal subgroup generated by $\{h^2\,;\, h \in F(Y)\}$. It follows that Φ

induces an isomorphism $\Psi : F(X)/N(X) \to F(Y)/N(Y)$. But $F(X)/N(X)$ is isomorphic to $\bigoplus_{x\in X}(\mathbb{Z}/2\mathbb{Z})_x$, while $F(Y)/N(Y)$ is isomorphic to $\bigoplus_{y\in Y}(\mathbb{Z}/2\mathbb{Z})_y$. Thus $\bigoplus_{x\in X}(\mathbb{Z}/2\mathbb{Z})_x$ is isomorphic to $\bigoplus_{y\in Y}(\mathbb{Z}/2\mathbb{Z})_y$, whence X and Y have the same cardinality. □

Remark 2.12. Proposition 2.11 implies that for every cardinal number n there exists, up to isomorphism, exactly one free group of rank n. We denote it by F_n.

Proposition 2.13 (Nielsen–Schreier)**.** *Any subgroup of a free group is a free group.*

For a proof see [LS77, Proposition 2.11].

Proposition 2.14. *The free group of rank two contains an isomorphic copy of F_k for k finite or $\aleph_0$.*

Idea of proof. Let x, y be the two generators of F_2. Let S be the subset consisting of all elements of F_2 of the form $y^n x y^{-n}$, for all $n \in \mathbb{N}$. The subgroup $< S >$ is isomorphic to the free group of rank $\aleph_0$. See [LS77, Proposition 3.1] for details.

2.4 Finitely generated groups

A *generating set* of a group G is a subset S of G such that every element of G can be expressed as the product of finitely many elements of S and their inverses. We also say that S generates G; the elements in S are called *generators*.

If a group has a finite generating set then it is called *finitely generated* or of *finite type*.

Nota bene: A finitely generated group is necessarily countable.

Remarks 2.15.
1. a quotient of a finitely generated group is finitely generated (by the images of the generators in the quotient);
2. if N is a normal subgroup in G and both N and G/N are finitely generated then G is finitely generated. Indeed take $\{n_1, .., n_k\}$ generating set for N, and $\{g_1 N, ..g_m N\}$ generating set for G/N. Then $\{g_i n_j \; ; \; i \in \{1, ..., m\}, j \in \{1, ..., k\}\}$ is a generating set for G.
3. A subgroup of a finitely generated group need not be finitely generated. See Proposition 2.14. Another example is the following. Take the group
$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \; ; \; a = 2^n \, , \, b = \frac{m}{2^k} \, , \, n, m, k \in \mathbb{Z} \right\}$$
which is generated by $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The subgroup composed of matrices with $a = 1$ is not finitely generated: for any finite set $b_1 = \frac{m_1}{2^{k_1}}, \dots, b_r = \frac{m_r}{2^{k_r}}$ all products of the corresponding matrices have elements $b \in \frac{1}{2^N}\mathbb{Z}$ with $N = \max\{k_1, \dots, k_r\}$.
4. A *finite index* subgroup of a finitely generated group is finitely generated. See the applications of Theorem 2.28.

Examples 2.16.
1. $(\mathbb{Z}, +)$ is finitely generated by both $\{1\}$ and $\{-1\}$. Also by $\{p, q\}$ if p and q are integers with $\gcd(p, q) = 1$.
2. $(\mathbb{Q}, +)$ is not finitely generated.
3. The group of permutations of $\mathbb{Z}$ with finite support (i.e. the group of one-to-one onto maps $f : \mathbb{Z} \to \mathbb{Z}$ such that f is identity outside a finite subset of $\mathbb{Z}$, endowed with the binary operation $\circ$) is *not* finitely generated. (*Exercise:* prove this statement.)

Example 2.17. The group $GL(n,\mathbb{Z})$ is generated by

$$s_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad s_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$s_3 = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad s_4 = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Proof. Step 1. Multiplication to the left with s_1 permutes rows cyclically, multiplication to the right does the same with columns. Multiplication to the left with s_2 swaps the first two rows, multiplication to the right does the same with columns. Therefore given any permutation σ on n elements there exists a product p of s_1 and s_2 such that multiplication to the left with p permutes rows following the permutation σ, multiplication to the right does the same with columns. This implies that any elementary matrix $E_{i,j} = I_n + N_{i,j}$, where $i \neq j$ and $N_{i,j}$ has the entry in the i–th row and j–th column 1 and all the other zero, can be written as a product of s_1, s_2, s_3. Also every diagonal matrix d_j which differs from the identity only in that the i–th entry in the diagonal is -1 is a product of s_1, s_2 and s_4. Note that with this notation $d_1 = s_4$.

Step 2. Now let g be an arbitrary element in $GL(n,\mathbb{Z})$. Let $a_1, ..., a_n$ be the entries of its first column. We prove that there exists a product p of powers of $s_1, ...s_4$ such that pg has the entries of its first column $1, 0, ..., 0$. We argue by induction on $k = |a_1| + \cdots + |a_n|$. Note that $k \geq 1$. If $k = 1$ then $(a_1, ..., a_n)$ is a permutation of $(1, 0, ..., 0)$ hence it suffices to take p the product of s_1, s_2 permuting the rows so as to obtain $1, 0, ..., 0$ in the first column.

Assume that the statement is true for al integers $i < k$, and we prove it for $k \geq 2$. Up to permuting rows and multiplying with $d_1 = s_4$ and d_2 we may assume that $a_1 > a_2 > 0$. Then $E_{1,2}d_2g$ has the entries on the first column $a_1 - a_2, -a_2, a_3, ...a_n$. By the induction argument there exists a product p of powers of $s_1, ...s_4$ such that $pE_{1,2}d_2g$ has the entries of its first column $1, 0, ..., 0$.

Step 3. It is easy to check that for every matrices $A, B \in GL(n-1,\mathbb{R})$ and rows $L = (l_1, ..., l_{n-1})$ and $M = (m_1, ..., m_{n-1})$

$$\begin{pmatrix} 1 & L \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 1 & M \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1 & M + LB \\ 0 & AB \end{pmatrix}.$$

Using this, an inductive argument and Step 2 it can be deduced that there exists a product p of powers of $s_1, ...s_4$ such that pg is upper triangular and with entries on the diagonal 1. It therefore suffices to prove that every upper triangular matrix is a product of powers of $s_1, ...s_4$. This can be done for instance by repeating the argument above with multiplications to the right. □

Exercise 2.18. Let G be a finitely generated group and let S be an infinite set of generators of G. Show that there exists $F \subset S$, F finite set generating G.

Exercise 2.19. An element g of the group G is a *non-generator* if for every generating set S of G containing g, $S \setminus \{g\}$ is still a generating set of G.

Prove that the set of non-generators forms a subgroup of G. This subgroup is called the *Frattini subgroup*.

Find the Frattini subgroup of $(\mathbb{Z}, +)$.

Find the Frattini subgroup of $(\mathbb{Z}^n, +)$. (*Hint:* You may use the fact that $Aut(\mathbb{Z}^n)$ is $GL(n, \mathbb{Z})$, and that the $GL(n, \mathbb{Z})$–orbit of e_1 is the set of vectors $(k_1, ..., k_n)$ in $\mathbb{Z}^n$ such that $\gcd(k_1, ..., k_n) = 1$.)

2.5 Cayley graphs

Finitely generated groups may be turned into geometric object as follows. Given a group G and a finite set of generators S such that $S^{-1} = S$ and $1 \notin S$, one can construct the *Cayley graph* of G with respect to S. This is a symmetric directed graph $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ such that

- its set of vertices is G;
- its set of oriented edges is (g, gs), with $s \in S \cup S^{-1}$.

We also call Cayley graph of G with respect to S the underlying non-oriented graph $\mathrm{Cayley}(G, S)$, i.e. the graph such that:

- its set of vertices is G;
- its set of edges is composed of all pairs of elements in G, $\{g_1, g_2\}$, such that $g_1 = g_2 s$, with $s \in S \cup S^{-1}$.

One can attach a *color* (*label*) from S to each oriented edge in $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$: the edge (g, gs) is labeled by s.

We suppose that every edge has length 1 and we endow $\mathrm{Cayley}(G, S)$ with the length metric. Its restriction to G is called *the word metric associated to* S and it is denoted by dist_S. See Figure 2 for the Cayley graph of the free group of rank two $\mathbb{F}_2 = \langle a, b \rangle$ with respect to $\{a, b\}$.

Notation 2.20. For an element $g \in G$ and a finite generating set S we denote $\mathrm{dist}_S(1, g)$ also by $|g|_S$. With this notation, $\mathrm{dist}_S(g, h) = |g^{-1}h|_S = |h^{-1}g|_S$.

Remark 2.21. A Cayley graph can be constructed also for an infinite set of generators. In this case the graph has infinite valence in each point.

Remark 2.22.

1. The group acts on itself by left multiplication:
$$G \times G \to G \,,\ (g, h) \mapsto gh \,.$$
This action extends to any Cayley graph, since edges are sent to edges. Both the action on the group and that on any Cayley graph are by isometry.
2. The action of the group on itself by right multiplication defines maps
$$R_g : G \to G \,,\ R_g(h) = hg$$
that are in general not isometries with respect to a word metric, but are at finite distance from the identity map:
$$\mathrm{dist}(\mathrm{id}(h), R_g(h)) = |g|_S \,.$$
This in particular implies that R_g is a $(1, |g|_S)$–quasi-isometry.

Exercise 2.23. Prove that the word metric on a group G associated to a generating set S containing together with every element its inverse may also be defined

1. either as the unique maximal left-invariant metric on G such that

$$\mathrm{dist}(1,s)=\mathrm{dist}(1,s^{-1})=1\,,\;\forall s\in S\,;$$

2. or by the following formula: $\mathrm{dist}(g,h)$ is the length of the shortest word w in the alphabet $S\cup S^{-1}$ such that $w=g^{-1}h$ in G.

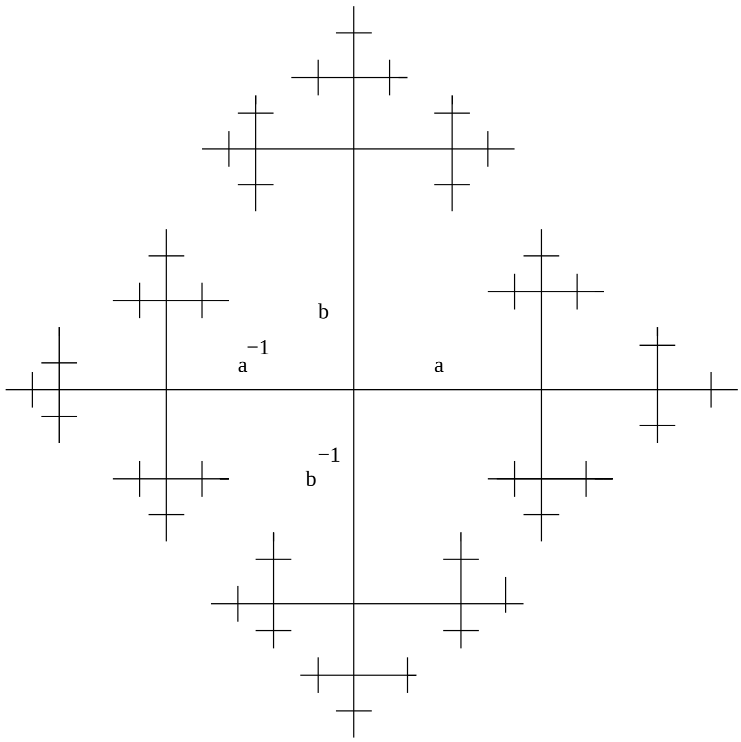

Figure 2: Cayley graph of $\mathbb{F}_2$.

Theorem 2.24. *A group is free if and only if it can act freely by automorphisms on a simplicial tree.*

A proof of the 'if' part can be found in [Ser80].

Exercise 2.25. 1. Prove that if S and $\bar{S}$ are two finite generating sets of G then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ are bi-Lipschitz equivalent, i.e. there exists $L>0$ such that

$$\frac{1}{L}\mathrm{dist}_S(g,g')\le\mathrm{dist}_{\bar{S}}(g,g')\le L\mathrm{dist}_S(g,g')\,,\forall g,g'\in G\,.$$

2. Prove that an isomorphism between two groups is a bi-Lipschitz map when the two groups are endowed with word metrics.

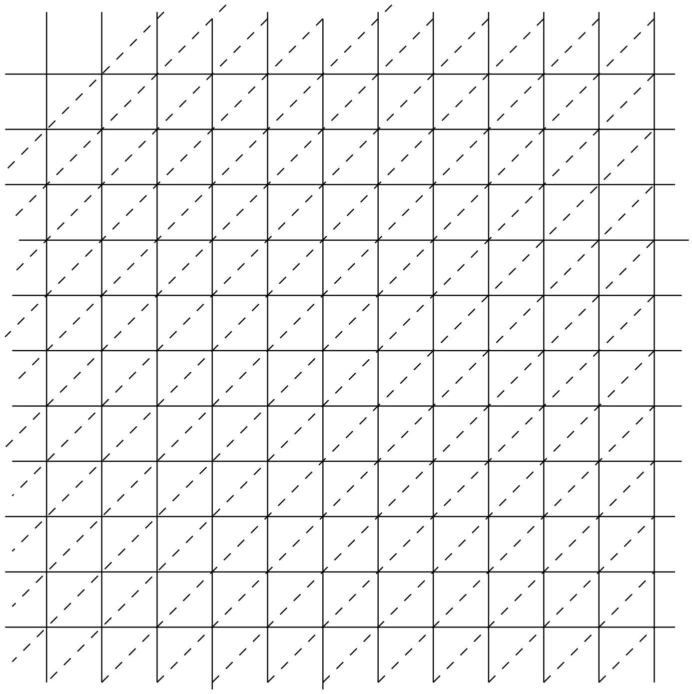

Figure 3: Cayley graph of $\mathbb{Z}^2$.

In Figure 3 are represented the Cayley graph of $\mathbb{Z}^2$ with set of generators $\{(1,0),(0,1)\}$ and the Cayley graph of $\mathbb{Z}^2$ with set of generators $\{(1,0),(1,1)\}$.

Exercise 2.26. Show that the Cayley graph of a finitely generated infinite group contains an isometric copy of $(\mathbb{R}, \|\cdot\|_e)$, i.e. a bi-infinite geodesic.

2.6 More examples of quasi-isometries

Example 2.27. All non-Abelian free groups of finite rank are quasi-isometric to each other. This follows from the fact that the Cayley graph of the free group of rank n with respect to a set of n generators and their inverses is the regular simplicial tree of valence $2n$.

Now all regular simplicial trees of valence at least 3 are quasi-isometric.

Proof. We denote by $\mathcal{T}_k$ the regular simplicial tree of valence k and we show that $\mathcal{T}_3$ is quasi-isometric to $\mathcal{T}_k$ for every $k \geq 4$.

We define the map $\mathfrak{q} : \mathcal{T}_3 \to \mathcal{T}_k$ as in Figure 4, sending all edges drawn in thin lines isometrically onto edges and all paths of length $k-3$ drawn in thick lines onto one vertex. The map $\mathfrak{q}$ thus defined is surjective and it satisfies the inequality

$$\frac{1}{k-2}\,\mathrm{dist}(x,y) - 1 \leq \mathrm{dist}(\mathfrak{q}(x),\mathfrak{q}(y)) \leq \mathrm{dist}(x,y)\,.$$

□

The main example, which partly justifies the interest in quasi-isometries, is given by the following result.

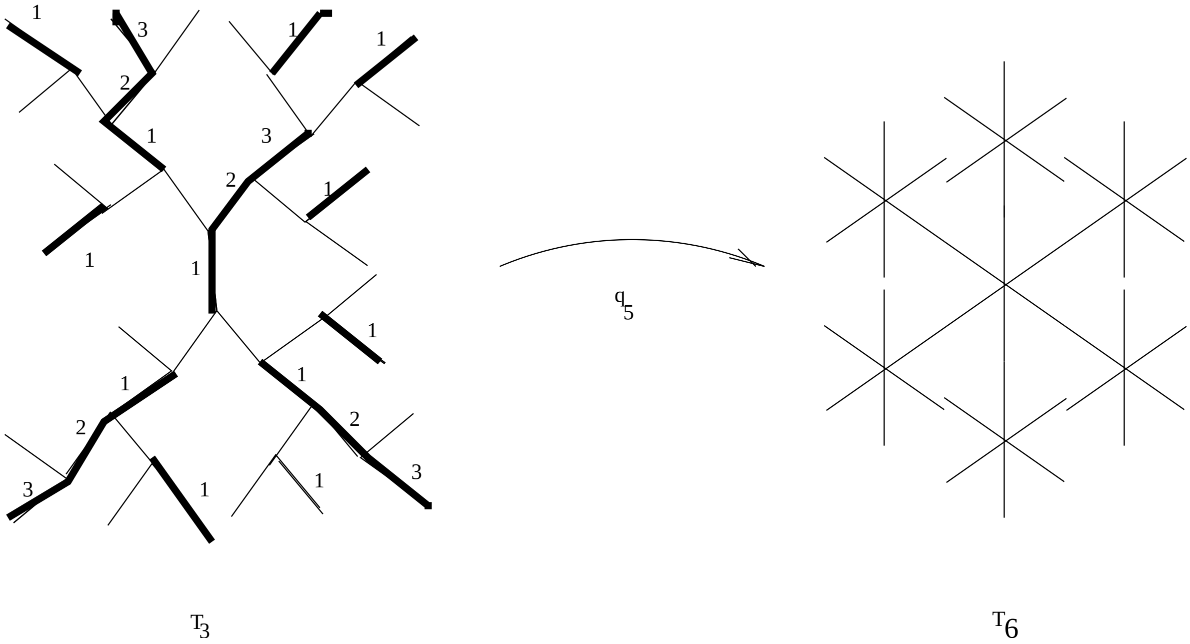

Figure 4: All regular simplicial trees are quasi-isometric.

Theorem 2.28. *[see for instance [dlH00], Theorem IV.B.23, or [BH99], Theorem I.8.10, p. 135]*

Let (X, dist) be a proper geodesic metric space (which is equivalent by Theorem 1.5 to X being a length metric space complete and locally compact) and let G be a group acting properly by isometries on X, with compact quotient $G\backslash X$. Then:

1. *the group G is finitely generated;*
2. *for any word metric dist_w on G and any point $x \in X$, the map $G \to X$ given by $g \mapsto gx$ is a quasi-isometry.*

Proof. We denote the orbit of a point $y \in X$ by Gy. Given a subset A in X we denote by GA the union of all orbits Ga with $a \in A$.

Step 1: The generating set.

The quotient space $G\backslash X$ has a canonical metric defined by

$$\mathrm{dist}(\bar{a}, \bar{b}) = \inf\{\mathrm{dist}(p, q)\ ;\ p \in Ga\,,\ q \in Gb\} = \inf\{\mathrm{dist}(a, q)\ ;\ q \in Gb\}\,.$$

The above infimum is attained, i.e. there exists $g \in G$ such that $\mathrm{dist}(\bar{a}, \bar{b}) = \mathrm{dist}(a, gb)$. Indeed take $g_0 \in G$ arbitrary, and let $R = \mathrm{dist}(a, gb)$. Then $\mathrm{dist}(\bar{a}, \bar{b}) = \inf\{\mathrm{dist}(a, q)\ ;\ q \in Gb \cap \overline{B}(a, R)\}$. Now for every $gb \in \overline{B}(a, R)$, $gg_0^{-1}\overline{B}(a, R) \cap \overline{B}(a, R) \neq \emptyset$. Since G acts properly, this implies that the set $Gb \cap \overline{B}(a, R)$ is finite, hence the last infimum is over a finite set, and it is attained.

With this, it is straightforward that $\mathrm{dist}(\bar{a}, \bar{b}) = 0 \Rightarrow \bar{a} = \bar{b}$, and symmetry is obvious.

Given $\bar{a}, \bar{b}, \bar{c}$ in $G\ X$ let $\mathrm{dist}(\bar{a}, \bar{b}) = \mathrm{dist}(ga, b)$ and $\mathrm{dist}(\bar{b}, \bar{c}) = \mathrm{dist}(b, hc)$. Then $\mathrm{dist}(ga, b) + \mathrm{dist}(b, hc) \geq \mathrm{dist}(ga, hc) \geq \mathrm{dist}(\bar{a}, \bar{c})$. Clearly the projection $X \to G\backslash X$ is a contraction. One can easily check that the topology induced by the metric dist on $G\backslash X$ coincides with the quotient topology.

By hypothesis $G\backslash X$ is compact therefore $G\backslash X$ is bounded. Let D be its (finite) diameter. Then for every $a \in X$ the ball $\overline{B}(a, D)$ intersects all orbits, in particular $G\overline{B}(a, D) = X$. For the rest of the proof we choose and fix an arbitrary point $x \in X$ and we denote $\overline{B}(x, D)$ simply by $\overline{B}$. Since X is proper, $\overline{B}$ is compact.

Define

$$S = \{s \in G \, ; \, s \neq 1 \, , \, s\overline{B} \cap \overline{B} \neq \emptyset\} \, .$$

Note that S is finite because the action of G is proper, and that $S^{-1} = S$ by definition.

Step 2: Outside the generating set.

Now consider $\inf\{\mathrm{dist}(\overline{B}, g\overline{B}) \, ; \, g \in G \setminus (S \cup \{1\})\}$. For some $g \in G \setminus (S \cup \{1\})$ the distance $\mathrm{dist}(\overline{B}, g\overline{B})$ is a positive constant R, by the definition of S. The set H of elements $h \in G$ such that $\mathrm{dist}(\overline{B}, h\overline{B}) \leq R$ is contained in the set $\{g \in G \, ; \, g\overline{B}(x, D+R) \cap \overline{B}(x, D+R) \neq \emptyset\}$, hence it is finite. Now $\inf\{\mathrm{dist}(\overline{B}, g\overline{B}) \, ; \, g \in G \setminus (S \cup \{1\})\} = \inf\{\mathrm{dist}(\overline{B}, g\overline{B}) \, ; \, g \in H \setminus (S \cup \{1\})\}$ and the latter infimum is over finitely many positive numbers, therefore there exists $h_0 \in H \setminus (S \cup \{1\})$ such that $\mathrm{dist}(\overline{B}, h_0\overline{B})$ realizes that infimum, which is therefore positive. Let then $2d$ be this infimum. By definition $\mathrm{dist}(\overline{B}, g\overline{B}) < 2d$ implies that $g \in S \cup \{1\}$.

Step 3: G is finitely generated.

Consider a geodesic $[x, gx]$ and $k = \left\lfloor \frac{\mathrm{dist}(x,gx)}{d} \right\rfloor$. Then there exists a finite sequence of points on the geodesic $[x, gx]$, $y_0 = x, y_1, ..., y_k, y_{k+1} = gx$ such that $\mathrm{dist}(y_i, y_{i+1}) \leq d$ for every $i \in \{0, ..., k\}$. For every $i \in \{1, ..., k\}$ let $h_i \in G$ be such that $y_i \in h_i\overline{B}$. We take $h_0 = 1$ and $h_{k+1} = g$. As $\mathrm{dist}(\overline{B}, h_i^{-1}h_{i+1}\overline{B}) = \mathrm{dist}(h_i\overline{B}, h_{i+1}\overline{B}) \leq \mathrm{dist}(y_i, y_{i+1}) \leq d$ it follows that $h_i^{-1}h_{i+1} = s_i \in S$, that is $h_{i+1} = h_i s_i$. Then $g = h_{k+1} = s_0 s_1 ... s_k$. We have thus proved that G is generated by S, consequently G is of finite type.

Step 4: The quasi-isometry.

Since all word metrics on G are bi-Lipschitz equivalent it suffices to prove (2) for the word metric dist_S, where S is the finite generating set found as above for the chosen arbitrary point x. The space X is contained in the $2D$–tubular neighborhood of the image Gx of the map defined in (2). It therefore remains to prove that the map is a quasi-isometric embedding. The previous argument proved that $|g|_S \leq k+1 \leq \frac{1}{d}\mathrm{dist}(x, gx) + 1$. Now let $|g|_S = m$ and let $w = s'_1 ... s'_m$ be a word in S such that $w = g$ in G. Then by the triangular inequality

$$\mathrm{dist}(x, gx) = \mathrm{dist}(x, s'_1 ... s'_m x) \leq \mathrm{dist}(x, s'_1 x) + \mathrm{dist}(s'_1 x, s'_1 s'_2 x) + \cdots + \mathrm{dist}(s'_1 ... s'_{m-1} x, s'_1 ... s'_m x)$$

$$= \sum_{i=1}^{m} \mathrm{dist}(x, s'_i x) \leq 2Dm = 2D|g|_S \, .$$

We have thus obtained that for any $g \in G$,

$$d\mathrm{dist}_S(1, g) - d \leq \mathrm{dist}(x, gx) \leq 2\mathrm{dist}_S(1, g) \, .$$

Sine both the word metric dist_S and the metric dist on X are left-invariant with respect to the action of G in the above 1 can be replaced by any element $h \in G$. □

Applications:

1. Given M a compact Riemannian manifold, let $\widetilde{M}$ be its universal covering and let $\pi_1(M)$ be its fundamental group. By Theorem 2.28, the group $\pi_1(M)$ is finitely generated, and the metric space $\widetilde{M}$ with the Riemannian metric is quasi-isometric to $\pi_1(M)$ with some

word metric. This can be clearly seen in the case when M is the n–dimensional flat torus $\mathbb{T}^n$. In this case $\widetilde{M}$ is $\mathbb{R}^n$ and $\pi_1(M)$ is $\mathbb{Z}^n$. They are quasi-isometric, as $\mathbb{R}^n$ is a thickening of $\mathbb{Z}^n$.

2. Two groups G_1 and G_2 acting properly discontinuously and with compact quotient by isometries on the same complete locally compact length metric space (X, dist_ℓ) are quasi-isometric.

3. Given a finitely generated group G and a finite index subgroup G_1 in it, G_1 is also finitely generated, and G, G_1 endowed with arbitrary word metrics are quasi-isometric. This may be seen as a particular case of the previous example, with $G_2 = G$ and X a Cayley graph of G.

4. Given a finite normal subgroup N in a finitely generated group G, G and G/N (both endowed with arbitrary word metrics) are quasi-isometric. This follows from Theorem 2.28 applied to the action of the group G on the Cayley graph of the group G/N.

   Thus, in arguments where we study behaviour of groups with respect to quasi-isometry, we can always replace a group with a finite index subgroup or with a quotient by a finite normal subgroup.

Open question 2.29 ([Gro93], p. 23, [BK02])**.** *If two finitely generated groups G and H endowed with word metrics are quasi-isometric, are they bi-Lipschitz equivalent ?*

Is this at least true when $H = G \times \mathbb{Z}/2\mathbb{Z}$? When H is a finite index subgroup in G ?

It is proved in [Why99] that two non-amenable groups are quasi-isometric if and only if they are bi-Lipschitz equivalent. This answers a question of Gromov [Gro93, § 1.A0]. It was previously proved that two free groups are bi-Lipschitz equivalent [Pap95]. The latter result has implications in L^2–cohomology.

2.7 Hyperbolic spaces in the sense of Gromov and real trees

There exist groups G acting on $\mathbb{H}^2$ properly and with compact quotient [BP92, Proposition B.3.1]. Hence they are finitely generated, and one may ask if their Cayley graph inherits any geometric features from $\mathbb{H}^2$. The property of hyperbolicity for geodesic metric spaces was introduced by M. Gromov partly inspired by the case of such groups.

Definition 2.30 (Rips' definition)**.** A geodesic metric space (X, d) is called δ–hyperbolic, $\delta > 0$, if every geodesic triangle $[a, b] \cup [b, c] \cup [c, a]$ in it is *δ–thin*: $\forall x \in [a, b]$ there exists $y \in [b, c] \cup [c, a]$ such that $\mathrm{dist}(x, y) \leq \delta$.

One may then introduce the following definition.

Definition 2.31. A *hyperbolic group* is a group with at least one Cayley graph hyperbolic.

We immediately see that (virtually) free groups are hyperbolic.
There are two problems with Definition 2.31:

- Is another Cayley graph of the same group still hyperbolic ?
- Is a group G acting in $\mathbb{H}^2$ properly and with compact quotient hyperbolic ?

Both can be summed up in the question:

Question 2.32. *If X, Y are two geodesic metric spaces, X is hyperbolic and Y is quasi-isometric to X, is Y hyperbolic ?*

The answer to this question is positive, and we shall see a proof of it later on. For the moment we consider another type of hyperbolic spaces.

Definition 2.33. A geodesic 0–hyperbolic metric space is called *real tree*.

Equivalently, it is a geodesic metric space such that any two points are connected by a unique topological arc.

Definition 2.34. Let T be a real tree and let p be a point in it. *The space of directions in* p, Σ_p, is defined as $\Re_p / \sim$, where

$$\Re_p = \{r : [0, a) \to T \mid a > 0,\ r \text{ isometry},\ r(0) = p\}$$

and

$$r_1 \sim r_2 \iff \exists\, \varepsilon > 0 \text{ such that } r_1|_{[0,\varepsilon)} \equiv r_2|_{[0,\varepsilon)}.$$

Definition 2.35. A *ramification point* of T is a point p such that card Σ_p is at least 3.

The number card Σ_p is called the *order of ramification* of the ramification point p.

The *order of ramification* of T is the maximal order of ramification in T.

Definition 2.36. A real tree is called *α–universal* if every real tree with order of ramification at most α can be isometrically embedded into it.

See [MNLGO92] for a study of universal trees. In particular the following holds.

Theorem 2.37 ([MNLGO92])**.** *For every cardinal number $\alpha > 2$ there exists an α–universal tree, and it is unique up to isometry.*

3 The Banach-Tarski paradox

Definition 3.1. Two subsets A, B in a metric space (X, dist) are *congruent* if there exists an isometry $\phi : X \to X$ such that $\phi(A) = B$.

Exercise 3.2. An *ideal triangle* in $\mathbb{H}^2$ is a triangle composed of three bi-infinite geodesics in $\mathbb{H}^2$ joining pairs of points in a triple $a, b, c \in \mathbb{R} \cup \{\infty\}$. Prove that any two ideal triangles in $\mathbb{H}^2$ are congruent.

Definition 3.3. Two sets A, B in a metric space X are *piecewise congruent* (or *equidecomposable*) if, for some $k \in \mathbb{N}$, they admit partitions $A = A_1 \sqcup ... \sqcup A_k$, $B = B_1 \sqcup ... \sqcup B_k$ such that for each $i \in \{1, ..., k\}$, the sets A_i and B_i are congruent.

Two subsets A, B in a metric space X are *countably piecewise congruent* (or *countably equidecomposable*) if they admit partitions $A = \bigsqcup_{n \in \mathbb{N}} A_n$, $B = \bigsqcup_{n \in \mathbb{N}} B_n$ such that for every $n \in \mathbb{N}$, the sets A_n and B_n are congruent.

Remark 3.4. We can see piecewise congruence as a stronger version of countably piecewise congruence by allowing for empty sets among A_n, B_n.

Exercise 3.5. Prove that (countably) piecewise congruence is an equivalence relation.

Definition 3.6. A set E in a metric space X is *paradoxical* if it admits a partition $E = X_1 \sqcup ... \sqcup X_k \sqcup Y_1 \sqcup ... \sqcup Y_m$ such that for some isometries $\varphi_1, ..., \varphi_k, \psi_1, ..., \psi_m$ of X, $\varphi_1(X_1) \sqcup ... \sqcup \varphi_k(X_k) = E$ and $\psi_1(Y_1) \sqcup ... \sqcup \psi_m(Y_m) = E$.

A set E in a metric space X is *countably paradoxical* if it admits a partition $E = \bigsqcup_{n\in\mathbb{N}} X_n \sqcup \bigsqcup_{m\in\mathbb{N}} Y_m$ such that for two sequences of isometries $(\varphi_n)_{n\in\mathbb{N}}, (\psi_m)_{m\in\mathbb{N}}$ of X, $\bigsqcup_{n\in\mathbb{N}} \varphi_n(X_n) = E$ and $\bigsqcup_{m\in\mathbb{N}} \psi_m(Y_m) = E$.

In the beginning the metric space (X, dist) will be $\mathbb{R}^n$ with the Euclidean metric. Recall the following well known facts.

Proposition 3.7.

1. *Any isometry ϕ of $\mathbb{R}^n$ is of the form $\phi(x) = Ax + b$, where $b \in \mathbb{R}^n$ and $A \in O(n)$.*
2. *The group $\mathrm{Isom}(\mathbb{R}^n)$ coincides with the semidirect product $\mathbb{R}^n \rtimes O(n)$, for the obvious action of $O(n)$ on $\mathbb{R}^n$.*

Sketch of proof of (1). If $\phi(0) = b$ then the isometry $\psi = T_{-b} \circ \phi$ fixes the origin, where T_{-b} is the translation of vector $-b$. Thus it suffices to prove that an isometry fixing the origin is a linear map in $O(n)$. Indeed:

- an isometry of $\mathbb{R}^n$ preserves straight lines, because these are bi-infinite geodesics;
- an isometry is a homogeneous map, i.e. $\psi(\lambda v) = \lambda\psi(v)$; this is due to the fact that λv is the unique vector on the same line through the origin as v, and at distance $|\lambda - 1|\|v\|$;
- an isometry map is an additive map, i.e. $\psi(a + b) = \psi(a) + \psi(b)$ because an isometry preserves parallelograms.

Thus $\psi(x) = Ax$ for some matrix A. The columns $\{C_1, ..., C_n\}$ of A are the images by ψ of the vectors of the canonical basis $\{e_1, ..., e_n\}$. Recall that the scalar product on $\mathbb{R}^n$, $\langle x, y\rangle$ is such that $\langle x, x\rangle = \|x\|^2$, and $\langle x, y\rangle = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2)$ by bi-linearity of the scalar product. Thus, an isometry fixing the origin preserves the scalar product. In particular, since $\langle e_i, e_j\rangle = \delta_{ij}$ it follows that $\langle C_i, C_j\rangle = \delta_{ij}$. This is equivalent to the fact that $A^T A = I_n$, since the (i, j)–entry of $A^T A$ is $\langle C_i, C_j\rangle$.

We conclude that $A \in O(n)$, hence $\phi(x) = T_b \circ \psi(x) = Ax + b$. □

Exercise 3.8. Prove (2). Note that $\mathbb{R}^n$ is identified to the group of translations via the map $b \mapsto T_b$.

Using work of Vitali and Hausdorff in an essential manner, Banach and Tarski proved the following.

Theorem 3.9 (the Banach-Tarski paradox [BT24])**.**

- *Any two bounded subsets with non-empty interior in an Euclidean space of dimension at least* 3 *are piecewise congruent.*
- *Any two bounded subsets with non-empty interior in $\mathbb{R}^n$ with $n \in \{1, 2\}$ are countably piecewise congruent.*

Corollary 3.10.

1. *An Euclidean ball is paradoxical in $\mathbb{R}^n, n \geq 3$, and countably paradoxical in $\mathbb{R}^n, n \in \{1, 2\}$.*
2. *In $\mathbb{R}^n, n \geq 3$, a ball is piecewise congruent with n copies of it, for any $n \in \mathbb{N}$.*
3. *A pea and the sun are piecewise congruent.*

Remark 3.11. The Banach-Tarski paradox emphasizes that it is impossible to find a finitely-additive measure defined on *all* subsets of an Euclidean space of dimension at least 3 that is invariant with respect to isometries and takes the value one on a unit cube. The main point in the theorem is that the congruent pieces A_i, B_i do not all have a well defined volume, i.e. they are not *Lebesgue measurable*.

3.1 Proof of the Banach–Tarski theorem.

We prove the Banach-Tarski theorem in the plane and in the 3-dimensional space, for A a unit ball and B a disjoint union of two isometric copies of the unit ball (this is sometimes called 'doubling the ball'). The general statement for two bounded subset of non-empty interior is derived from the doubling of a ball by using the Banach–Bernstein-Schroeder theorem (see [Wag85]). The general statement in $\mathbb{R}^n, n \geq 3$, can be easily either derived from the statement for $n = 3$, or proved directly by adapting the proof in dimension 3.

The first step in the proof is common to all dimensions.

Step 1: the unit sphere $\mathbb{S}^n$ is piecewise congruent to $\mathbb{S}^n \setminus C$, where C is any countable set, and $n = 1, 2$.

We first prove that there exists a rotation ρ around the origin such that for any integer $n \geq 1$, $\rho^n(C) \cap C = \emptyset$. This is obvious in the plane (only a countable set of rotations do not satisfy this).

In the space we first select a line ℓ through the origin such that its intersection with $\mathbb{S}^2$ is not in C. Such a line exists because the set of lines through the origin containing points in C is countable. Then we look for a rotation ρ_θ of angle θ around ℓ such that for any integer $n \geq 1$, $\rho_\theta^n(C) \cap C = \emptyset$. Indeed take A the set of angles α such that the rotation of angle α around ℓ sends a point in C to another point in C. There are countably many such angles, therefore the set $A' = \bigcup_{n \geq 1} \frac{1}{n} A$ is likewise countable. Thus we may choose an angle $\theta \notin A'$.

Take $\mathcal{O} = \bigcup_{n \geq 0} \rho_\theta^n(C)$ and decompose $\mathbb{S}^2 = \mathcal{O} \sqcup (\mathbb{S}^2 \setminus \mathcal{O})$. Then $\mathbb{S}^2$ is piecewise congruent to $\rho_\theta(\mathcal{O}) \sqcup (\mathbb{S}^2 \setminus \mathcal{O}) = (\mathcal{O} \setminus C) \sqcup (\mathbb{S}^2 \setminus \mathcal{O}) = \mathbb{S}^2 \setminus C$.

3.1.A Proof of the Banach–Tarski theorem in the plane.

Step 2 (using the axiom of choice): the unit circle is countably paradoxical.

Let α be an irrational number and let R be the counter-clockwise rotation around the origin of angle $2\pi\alpha$. Then the map $m \mapsto R^m$ is an injective homomorphism $\mathbb{Z} \to SO(2)$. Via this homomorphism $\mathbb{Z}$ acts on the unit circle $\mathbb{S}^1$. *According to the axiom of choice* we can choose one point in every orbit of $\mathbb{Z}$. Let D be the set composed of all these points. Then $\mathbb{Z}D = \mathbb{S}^1$, and every orbit intersects D exactly once.

Since $\mathbb{Z}$ decomposes as $2\mathbb{Z} \sqcup (2\mathbb{Z}+1)$, the unit circle decomposes as $2\mathbb{Z}D \sqcup (2\mathbb{Z}+1)D$. Now for each $X_n = R^{2n}D$ consider the isometry $\varphi_n = R^{-n}$, and for each $Y_n = R^{2n+1}D$ consider the isometry $\psi_n = R^{-n-1}$. Clearly $\mathbb{S}^1 = \bigsqcup_{n \in \mathbb{Z}} \varphi_n(X_n)$ and $\mathbb{S}^1 = \bigsqcup_{n \in \mathbb{Z}} \psi_n(Y_n)$.

Step 3: the unit disk is countably paradoxical.

Step 1 and the fact that the unit disk $\mathbb{D}^2$ without the origin O can be written as the set $\{\lambda x \; ; \; \lambda \in (0,1] \, , \, x \in \mathbb{S}^1\}$ implies that $\mathbb{D}^2 \setminus \{O\}$ is countably paradoxical. Thus, it suffices to prove that $\mathbb{D}^2 \setminus \{O\}$ is piecewise congruent with $\mathbb{D}^2$. Take $\mathbb{S}^1\left(\left(\frac{1}{2}, 0\right), \frac{1}{2}\right)$ the unit circle of center $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. For simplicity we denote it by $\mathbb{S}_{1/2}$. Then $\mathbb{D}^2 \setminus \{O\} = \mathbb{D}^2 \setminus \mathbb{S}_{1/2} \sqcup \mathbb{S}_{1/2} \setminus \{O\}$. According to Step 1, $\mathbb{S}_{1/2} \setminus \{O\}$ is piecewise congruent with $\mathbb{S}_{1/2}$, hence $\mathbb{D}^2 \setminus \{O\}$ is piecewise congruent with $\mathbb{D}^2 \setminus \mathbb{S}_{1/2} \sqcup \mathbb{S}_{1/2} = \mathbb{D}^2$.

□

Remark 3.12 (Stronger result). Instead of the splitting $\mathbb{Z} = 2\mathbb{Z} \sqcup (2\mathbb{Z}+1)$ of $\mathbb{Z}$ into two 'copies' of itself, we might consider a splitting of $\mathbb{Z}$ into infinitely countably many 'copies' of itself. Indeed the subsets $\mathbb{Z}^{(k)} = 2^k\mathbb{Z} + 2^{k-1}$, $k \in \mathbb{N}$, compose a partition of $\mathbb{Z}$. This allows to prove, following the same outline as above, that a unit disk is countably piecewise congruent with countably many copies of itself.

Proof. As in Step 2, we write $\mathbb{S}^1 = \mathbb{Z}D = \bigsqcup_{k\in\mathbb{N}} \mathbb{Z}^{(k)}D$. The idea is to move by isometries the copies of D in $\mathbb{Z}^{(k)}D$ so as to compose the k-th copy of the unit sphere. Indeed, if for the set $X_{k,m} = R^{2^k m+2^{k-1}}D$ we consider the isometry $\phi_{k,m} = T_{(2k,0)} \circ R^{-2^k m-2^{k-1}+m}$ then $\bigsqcup_{m\in\mathbb{Z}} \phi_{k,m}(X_{k,m})$ is equal to $\mathbb{S}^1((2k, 0), 1)$.

Thus $\mathbb{S}^1$ is countably piecewise congruent with $\bigsqcup_{k\in\mathbb{N}} \mathbb{S}^1((2k, 0), 1)$. This extends to the corresponding disks deprived of their respective centers. In Step 3 we proved that a disk without origin is piecewise congruent to the full disk. This allows to finish the argument. □

3.1.B Proof of the Banach–Tarski theorem in the space.

We now explain the proof of the Banach–Tarski theorem for A the unit ball in $\mathbb{R}^3$ and B the disjoint union of two unit balls in $\mathbb{R}^3$.

Step 2: a paradoxical decomposition for the free group of rank 2.

Let F_2 be a free group of rank 2 with generators a, b. Given u a reduced word in a, b, a^{-1}, b^{-1}, denote by $\mathcal{W}_u$ the set of reduced words in a, b, a^{-1}, b^{-1} with prefix u.

Then

$$F_2 = \{1\} \sqcup \mathcal{W}_a \sqcup \mathcal{W}_{a^{-1}} \sqcup \mathcal{W}_b \sqcup \mathcal{W}_{b^{-1}}$$

but also $F_2 = L_a\mathcal{W}_{a^{-1}} \sqcup \mathcal{W}_a$, and $F_2 = L_b\mathcal{W}_{b^{-1}} \sqcup \mathcal{W}_b$, where L_x is defined as in the proof of Proposition 2.1. We slightly modify the above partition in order to include $\{1\}$ into one of the other four subsets. Consider the following modifications of $\mathcal{W}_a$ and $\mathcal{W}_{a^{-1}}$:

$$\mathcal{W}'_a = \mathcal{W}_a \setminus \{a^n\ ;\ n \geq 1\} \text{ and } \mathcal{W}'_{a^{-1}} = \mathcal{W}_{a^{-1}} \sqcup \{a^n\ ;\ n \geq 0\}\,.$$

Then

$$F_2 = \mathcal{W}'_a \sqcup \mathcal{W}'_{a^{-1}} \sqcup \mathcal{W}_b \sqcup \mathcal{W}_{b^{-1}} \tag{7}$$

and

$$F_2 = L_a\mathcal{W}'_{a^{-1}} \cup \mathcal{W}'_a\,.$$

Step 3: a paradoxical decomposition for the unit sphere (using the axiom of choice).

According to the Example 2.8 we may see F_2 as a subgroup in the group $SO(3)$ of linear isometries of $\mathbb{R}^3$. For every word w in F_2 we denote by R_w the corresponding rotation.

Let C be the set of intersections of $\mathbb{S}^2$ with the axes of the rotations R_w with $w \in F_2$. Note that if $\{x, y\}$ is the intersection of one such axis with $\mathbb{S}^2$ then $C = F_2\{x, y\}$, i.e. C is composed of the F_2 orbits of x and of y. The set C is countable, and by Step 1, $\mathbb{S}^2$ is piecewise congruent with $\mathbb{S}^2 \setminus C$. The set $\mathbb{S}^2 \setminus C$ is a disjoint union of orbits of F_2. *According to the axiom of choice* we can choose one point in every orbit. Let D be the set composed of all these points. Then $F_2D = \mathbb{S}^2 \setminus C$, and every orbit intersects D exactly once. The removal of the set C ensures that the action of F_2 is free, i.e. no element of F_2 fixes a point, that is all orbits are copies of F_2.

By Step 2, $F_2 = \mathcal{W}'_a \sqcup \mathcal{W}'_{a^{-1}} \sqcup \mathcal{W}_b \sqcup \mathcal{W}_{b^{-1}}$. This defines a splitting

$$\mathbb{S}^2 \setminus C = F_2D = \mathcal{W}'_aD \sqcup \mathcal{W}'_{a^{-1}}D \sqcup \mathcal{W}_bD \sqcup \mathcal{W}_{b^{-1}}D\,. \tag{8}$$

The fact that the subsets composing the union in (8) are pairwise disjoint is precisely due to the fact that the action of F_2 is free.

The set $\mathbb{S}^2 \setminus C$ is piecewise congruent with

$$\mathcal{W}'_a D \sqcup R_a \mathcal{W}'_{a^{-1}} D \sqcup T_{(3,0,0)} (\mathcal{W}_b D) \sqcup T_{(3,0,0)} \circ R_b (\mathcal{W}_{b^{-1}} D) = \mathbb{S}^2 \setminus C \sqcup T_{(3,0,0)} \left(\mathbb{S}^2 \setminus C\right) ,$$

where $T_{(3,0,0)}$ denotes the translation of vector $(3, 0, 0)$.

This and Step 1 imply that $\mathbb{S}^2$ is piecewise congruent with $\mathbb{S}^2 \sqcup T_{(3,0,0)}\mathbb{S}^2$.

Step 4: a paradoxical decomposition for the unit ball.

The argument is very similar to the last step in the 2-dimensional case.

Step 3 and the fact that the unit ball $\mathbb{B}^3$ without the origin O can be written as the set $\{\lambda x \; ; \; \lambda \in (0, 1] , x \in \mathbb{S}^2\}$ implies that $\mathbb{B}^3 \setminus \{O\}$ is piecewise congruent with $\mathbb{B}^3 \setminus \{O\} \sqcup T_{(3,0,0)} \left(\mathbb{B}^3 \setminus \{O\}\right)$. Thus, it remains to prove that $\mathbb{B}^3 \setminus \{O\}$ is piecewise congruent with $\mathbb{B}^3$. We denote by $\mathbb{S}_{1/2}$ the sphere of center $\left(\frac{1}{2}, 0, 0\right)$ and radius $\frac{1}{2}$. Then $\mathbb{B}^3 \setminus \{O\} = \mathbb{B}^3 \setminus \mathbb{S}_{1/2} \sqcup \mathbb{S}_{1/2} \setminus \{O\}$. According to Step 1, $\mathbb{S}_{1/2} \setminus \{O\}$ is piecewise congruent with $\mathbb{S}_{1/2}$, hence $\mathbb{B}^3 \setminus \{O\}$ is piecewise congruent with $\mathbb{B}^3 \setminus \mathbb{S}_{1/2} \sqcup \mathbb{S}_{1/2} = \mathbb{B}^3$.

□

Remark 3.13. Banach and Tarski's proof relies on Hausdorff's paradox, discovered several years earlier. Inspired by Hausdorff's argument, R. M. Robinson proved in [Rob47] that the minimal number of pieces in a paradoxical decomposition of the unit 3-dimensional ball is five, and answered thus a question of von Neumann.

Remark 3.14.

1. An iteration of the above construction proves that it is possible to obtain k copies of a ball in the Euclidean n–space from one, for any integers $n \geq 3$ and $k \geq 1$.
2. The free group F_2 of rank 2 contains a free subgroup of countably infinite rank, by Proposition 2.14. This and a proof similar to the previous yields that the unit sphere S^{n-1} is countably piecewise congruent to countably many copies of S^{n-1}.
3. It can be proved that the unit sphere S^{n-1} can be partitioned into $2^{\aleph_0}$ pieces, so that each piece is piecewise congruent to S^{n-1}. [Wag85].

3.2 Versions of the axiom of choice: from the Banach–Tarski paradox to an extension of the notion of limit

We first recall that the *Zermelo-Fraenkel axioms (ZF)* compose a list of axioms which are the basis of axiomatic set theory in its standard form. See for instance [Kun80] or [Jec03]. The Banach-Tarski paradox is neither provable nor disprovable in ZF only: it is impossible to prove that a unit ball in $\mathbb{R}^3$ is paradoxical in ZF, it is also impossible to prove it is not paradoxical. An extra axiom is needed, the axiom of choice (AC), first formulated by E. Zermelo in [Zer04]. According to work of K. Gödel and P. Cohen, the axiom of choice is logically independent of the other axioms of Zermelo-Fraenkel (i.e. neither it nor its negation can be proven in ZF).

The axiom of choice can be seen as a rule of building sets out of other sets.

Given a non-empty collection $\mathcal{S}$ of non-empty sets, a *choice function* defined on $\mathcal{S}$ is a function f such that for every set A in $\mathcal{S}$, $f(A)$ is an element of A. A choice function on $\mathcal{S}$ can be viewed as an element of the Cartesian product $\prod_{A\in\mathcal{S}} A$.

Axiom of choice

On any non-empty collection of non-empty sets one can define a choice function. Equivalently, an arbitrary Cartesian product of non-empty sets is non-empty.

Remark 3.15. If $\mathcal{S} = \{A\}$ then the existence of f follows from the fact that A is non-empty. If $\mathcal{S}$ is finite the existence of a choice function can be proved by induction. Thus, if the collection $\mathcal{S}$ is finite then the existence of a choice function follows from ZF.

Remark 3.16. Assuming ZF, the Axiom of choice is equivalent to each of the following statements:

1. *Zorn's lemma.*

2. Every vector space has a basis.

3. Every ideal in a unitary ring is contained in a maximal ideal.

4. If A is a subset in a topological space X and B is a subset in a topological space Y the closure of $A \times B$ in $X \times Y$ is equal to the product of the closure of A in X with the closure of B in Y.

5. (*Tychonoff's theorem:*) If $(X_i)_{i\in I}$ is a collection of non-empty compact topological spaces then $\prod_{i\in I} X_i$ is compact.

Remark 3.17. The following statements require the Axiom of choice:

1. every union of countably many countable sets is countable;

2. The Nielsen–Schreier theorem: every subgroup of a free group is free.

In ZF, we have the following irreversible sequence of implications:

Axiom of choice $\Rightarrow$ Ultrafilter Lemma $\Rightarrow$ the Hahn–Banach extension theorem.

The first implication is well known, it was proved to be irreversible in [Hal64]. The second implication is proved in ([ŁRN51], [Lux62], [Lux67], [Lux69]). Its irreversibility is proved in [Pin72] and [Pin74].

So the Hahn–Banach extension theorem can be seen as the analyst's Axiom of Choice, in a weaker version. We recall the theorem here.

Theorem 3.18 (Hahn–Banach [Roy68])**.** *Let V be a real vector space, U a subspace of it, and $f : U \to \mathbb{R}$ linear function. Let $p : V \to \mathbb{R}$ be a map with the following properties:*

$$p(\lambda x) = \lambda p(x) \text{ and } p(x+y) \leq p(x) + p(y)\,,\ \forall x, y \in V,\ \lambda \in [0, +\infty)\,,$$

such that $f(x) \leq p(x)$ for every $x \in U$. Then there exists a linear extension of f, $\overline{f} : V \to \mathbb{R}$ such that $\overline{f}(x) \leq p(x)$ for every $x \in V$.

Work of M. Foreman & F. Wehrung [FW91] and J. Pawlikowski [Paw91] shows that the Banach–Tarski paradox can be proved assuming ZF and the Hahn–Banach theorem.

Since we use the Ultrafilter Lemma in an essential way later on, we also recall it here.

Definition 3.19. A *filter* $\mathcal{F}$ on a set I is a collection of subsets of I satisfying the following conditions:

(F_1) $\emptyset \notin \mathcal{F}$;

(F_2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(F_3) If $A \in \mathcal{F}$, $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Exercise 3.20. Given an infinite set I, prove that the collection of all complementaries of finite sets is a filter on I. This filter is called *the Fréchet filter*.

Definition 3.21. An *ultrafilter* on a set I is a filter $\mathcal{U}$ on I which is a maximal element in the ordered set of all filters on I with respect to the inclusion. An ultrafilter can also be defined as a collection of subsets of I satisfying the conditions (F_1), (F_2), (F_3) defining a filter and the additional condition:

$$(F_4) \text{ For every } A \subseteq I \text{ either } A \in \mathcal{U} \text{ or } I \setminus A \in \mathcal{U}.$$

Exercise 3.22. Given a set I, take a point $x \in I$ and consider the collection $\mathcal{U}_x$ of subsets of I containing x. Prove that $\mathcal{U}_x$ is an ultrafilter on I.

Definition 3.23. An ultrafilter as in Exercise 3.22 is called *a principal (or atomic) ultrafilter.* A filter that cannot be defined in such a way is called a *a non-principal (or non-atomic) ultrafilter.*

Proposition 3.24. *An ultrafilter on an infinite set I is non-principal if and only if it contains the Fréchet filter.*

Proof. We will in fact prove the equivalence between the negations of the two statements.

A principal ultrafilter $\mathcal{U}$ on I defined by a point x contains $\{x\}$ hence by (F_4) it does not contain $I \setminus \{x\}$ which is an element of the Fréchet filter.

Let now $\mathcal{U}$ be an ultrafilter that does not contain the Fréchet filter. This and property (F_4) implies that it contains a finite subset F of I.

If $F \cap \bigcap_{A \in \mathcal{U}} A = \emptyset$ then there exist $A_1, ..., A_n \in \mathcal{U}$ such that $F \cap A_1 \cap \cdots \cap A_n = \emptyset$. This and property (F_2) contradict property (F_1).

It follows that $F \cap \bigcap_{A \in \mathcal{U}} A = F_1 \neq \emptyset$ in particular given an element $x \in F_1$, $\mathcal{U}$ is contained in the principal ultrafilter $\mathcal{U}_x$. The maximality of $\mathcal{U}$ implies that $\mathcal{U} = \mathcal{U}_x$. □

The Ultrafilter Lemma: Every filter on a set I is a subset of some ultrafilter on I.

In ZF, the Axiom of Choice is equivalent to Zorn's Lemma, and the latter clearly implies the Ultrafilter Lemma.

Definition 3.25. An equivalent way of defining *an ultrafilter on a set I* is as a finitely additive measure ω defined on $\mathcal{P}(I)$, taking only values zero and one and such that $\omega(I) = 1$. Indeed, ω satisfies the previous properties if and only if it is the characteristic function $\mathbf{1}_{\mathcal{U}}$ of a collection $\mathcal{U}$ of subsets of I which is an ultrafilter.

Note that for an atomic ultrafilter $\mathcal{U}_x$ defined as in Example 3.22 the corresponding measure is the Dirac measure δ_x.

Definition 3.26. A *non-principal ultrafilter on a set I* is a finitely additive measure $\omega : \mathcal{P}(I) \to \{0, 1\}$ such that $\omega(I) = 1$ and $\omega(F) = 0$ for every finite subset F of I.

Exercise 3.27. Prove the equivalence between Definitions 3.21 and 3.25, and between Definitions 3.23 and 3.26.

Remark 3.28. If $\omega(A_1 \sqcup \cdots \sqcup A_n) = 1$, then there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\omega(A_{i_0}) = 1$ and $\omega(A_j) = 0$ for every $j \neq i_0$.

We now explain how existence of non-principal ultrafilters implies Hahn–Banach in a particular case: the one when V is the real vector space of bounded sequences of real numbers $\mathbf{x} = (x_n)$, U is the subspace of convergent sequences of real numbers, p is the sup-norm $\|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$ and $f : U \to \mathbb{R}$ is the limit function, i.e. $f(\mathbf{x}) = \lim_{n\to\infty} x_n$. In other words we show how using a non-principal ultrafilter one can extend the notion of limit from convergent sequences to bounded sequences.

Definition 3.29. Given an ultrafilter ω on $\mathbb{N}$ (in the sense of Definition 3.25) and a sequence (x_n) in a topological space, one can define the *ω–limit* $\lim_\omega x_n$ of the sequence as an element x such that for every open set O containing x,

$$\omega\left(\{n \in \mathbb{N} \mid x_n \in O\}\right) = 1\,.$$

Proposition 3.30. *If (x_n) is contained in a compact metric space, its ω–limit always exists.*

Proof. The closure K of the set of elements $\{x_n\ ;\ n \in \mathbb{N}\}$ is also compact. Take $\epsilon > 0$ arbitrary. Since $K \subset \bigcup_{y\in K} B(y,\epsilon)$ it follows that $K \subset \bigcup_{y\in F} B(y,\epsilon)$, where F is a finite subset in K. In particular $\mathbb{N}$ decomposes into a union (not necessarily disjoint) $\bigcup_{y\in F} \mathbb{N}_y$, where $\mathbb{N}_y = \{n \in \mathbb{N}\ ;\ x_n \in B(y,\epsilon)\}$. Since $1 = \omega(\mathbb{N}) = \omega(\bigcup_{y\in F} \mathbb{N}_y)$ it follows that there exists $y \in F$ such that $\omega(\mathbb{N}_y) = 1$, that is "$x_n \in B(y,\epsilon)$ ω-almost surely ".

Thus for every integer $k \in \mathbb{N}$, by applying the above to $\epsilon = \frac{1}{k}$ one obtains a point y_k such that $x_n \in B(y_k, 1/k)$ for ω-almost every n. Now y_k is in a compact metric space hence it contains a subsequence $(y_{\phi(k)})$ converging to a point y. We claim that y is an ω–limit of (x_n). Indeed take $\varepsilon > 0$. For $k \geq k_0$, $y_{\phi(k)} \in B(y, \varepsilon/2)$. Possibly by increasing k_0 we may ensure that $\frac{1}{\phi(k)} \leq \frac{\varepsilon}{2}$ for every $k \geq k_0$. Then for some fixed $k \geq k_0$, ω-almost surely $x_n \in B(y_{\phi(k)}, 1/\phi(k)) \subseteq B(y_{\phi(k)}, \frac{\varepsilon}{2}) \subseteq B(y,\varepsilon)$. □

Remark 3.31. It is not difficult to see that the ω–limit of a sequence is a limit of a converging subsequence of that sequence. Thus, an ultrafilter is a device to select a point of accumulation for any sequence contained in a compact metric space, in a coherent manner.

Note that when the ultrafilter is principal, that is $\omega = \delta_{n_0}$ for some $n_0 \in \mathbb{N}$, the δ_{n_0}–limit of a sequence (x_n) is simply the element x_{n_0}, so not very interesting. Thus, when considering ω–limits we shall always choose the ultrafilter ω to be non-principal.

Remark 3.32. Recall that when we have a countable collection of sequences $\mathbf{x}^{(k)} = \left(x_n^{(k)}\right)_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, all contained in a compact metric space, we can select a set of indices $I \subset \mathbb{N}$ such that for every $k \in \mathbb{N}$ the subsequence $\left(x_i^{(k)}\right)_{i\in I}$ is convergent. This is achieved by the diagonal procedure. The ω–limit allows, in some sense, to do the same for an uncountable collection of sequences. Thus it can be seen as an uncountable version of the diagonal procedure.

Exercise 3.33. Prove that the ω–limit of a sequence is a limit of a converging subsequence.

With such a tool at hand, which makes almost any reasonable sequence converge, we shall be able to define, for a given metric space (X, dist), an image of it seen from infinitely far away. More details on this will appear in Section 6.

For more details on filters and ultrafilters see [Bou65, §I.6.4].

4 Amenable groups.

4.1 Definition and properties

Let G be a group acting on a set X. We define *G–congruent* pairs of subsets of X, and *G–(countably) piecewise congruent* pairs of subsets, as well as *G-paradoxical sets* as in Definitions 3.1, 3.3 and 3.6 with isometries replaced by transformations in G. The G–(countably) piecewise congruence is still an equivalence relation.

Example 4.1. We proved in Section 3.1.B that the free group F_2 is paradoxical with respect to its own action on itself to the left.

John von Neumann [vN29] studied the properties of the actions of groups that make a paradoxical decomposition possible (like isometries of $\mathbb{R}^n$ with $n \geq 3$). He defined the class of *amenable groups*, for which no paradoxical decompositions exist.

Definition 4.2. An *algebra* of subsets of a space X is a non-empty collection $\mathcal{A}$ of subsets of X such that:

1. $\emptyset$ and X are in $\mathcal{A}$;

2. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$;

3. $A \in \mathcal{A} \Rightarrow A^c = X \setminus A \in \mathcal{A}$.

Definition 4.3. A *finitely additive probability (f.a.p.) measure μ on an algebra* $\mathcal{A}$ of subsets of X is a function $\mu : \mathcal{A} \to [0, 1]$ such that $\mu(X) = 1$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$.

Let G be a group acting on X so that $gA \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $g \in G$. If $m(gA) = m(A)$ for any $g \in G$ and $A \in \mathcal{R}$, then m is called a *G–invariant f.a.p. measure.*

An immediate consequence of the above is the following. For any two sets A, B, $m(A \cup B) = m((A \setminus B) \sqcup (A \cap B) \sqcup (B \setminus A)) = m(A \setminus B) + m(A \cap B) + m(B \setminus A) \leq m(A) + m(B)$.

Remark 4.4. In some texts the same notion is called simply 'measure'. We prefer the terminology above, since in other texts by 'measure' is meant a countably additive measure.

Definition 4.5. A group G is *amenable* if there exists a G-invariant f.a.p. measure μ on $\mathcal{P}(G)$.

Example 4.6. Finite groups are amenable. Indeed, take G finite and define $\mu : \mathcal{P}(G) \to [0, 1]$ by $\mu(A) = \frac{|A|}{|G|}$.

Example 4.7. The free group of rank two is not amenable.

Proof. Assume there exists a left-invariant f.a.p. measure μ on F_2. With the decomposition defined in (7) we have that $1 = \mu(F_2) = \mu(\mathcal{W}'_a \sqcup \mathcal{W}'_{a^{-1}} \sqcup \mathcal{W}_b \sqcup \mathcal{W}_{b^{-1}}) = \mu(\mathcal{W}'_a) + \mu(\mathcal{W}'_{a^{-1}}) + \mu(\mathcal{W}_b) + \mu(\mathcal{W}_{b^{-1}}) = \mu(\mathcal{W}'_a) + \mu(L_a \mathcal{W}'_{a^{-1}}) + \mu(\mathcal{W}_b) + \mu(L_b \mathcal{W}_{b^{-1}}) = \mu(\mathcal{W}'_a \sqcup L_a \mathcal{W}'_{a^{-1}}) + \mu(\mathcal{W}_b \sqcup L_b \mathcal{W}_{b^{-1}}) = 2$, which is absurd. □

Convention 4.8. Throughout the paper we denote by $\mathbf{1}_A$ the characteristic function of a subset A in a space X, i.e. the function $\mathbf{1}_A : X \to \{0, 1\}$, $\mathbf{1}_A(x) = 1$ if and only if $x \in A$.

Given a f.a.p. measure μ on G one can apply the standard construction of integrals (see [Rud, Chapter 1] or [Roy68, Chapter 11]) and define, for any function $f : G \to \mathbb{C}$, $m(f) = \int f \, d\mu$. In particular, given the set $\ell^\infty(G)$ of bounded functions on G the integral defines a linear functional $m : \ell^\infty(G) \to \mathbb{C}$ such that:

(M1) if f take values in $[0, \infty)$ then $m(f) \geq 0$;

(M2) $m(\mathbf{1}_G) = 1$.

Such a map is called a *mean* on G.

The group G has a canonical left action on $\ell^\infty(G)$ defined by $g \cdot f(x) = f(g^{-1}x)$, that is $g \cdot f = f \circ L_{g^{-1}}$. A mean is called *left-invariant* if $m(g \cdot f) = m(f)$ for every $f \in \ell^\infty(G)$ and $g \in G$. Note that if the measure μ is left-invariant then m is left-invariant.

Conversely, given a (left-invariant) mean, one can define a (left-invariant) f.a.p. measure by $\mu(A) = m(\mathbf{1}_A)$.

Exercise 4.9. Prove that μ thus defined is a f.a.p. measure and that m left-invariant implies μ left-invariant.

We have thus proved the following:

Proposition 4.10. *A group G is amenable if and only if it admits a left-invariant mean.*

Remark 4.11. (a) In the above, left-invariance can be replaced by right-invariance.

(b) Moreover, both can be replaced by two-sided invariance.

Proof. (a) It suffices to define $\mu_r(A) = \mu(A^{-1})$ and $m_r(f) = m(f_1)$, where $f_1(x) = f(x^{-1})$.

(b) Let μ be a left-invariant f.a.p. measure and μ_r the right invariant measure in (a). Then for every $A \subseteq X$ define

$$\nu(A) = \int \mu(Ag^{-1})d\mu_r(g)\,.$$

□

Lemma 4.12. *If an amenable group G acts on a space X then there exists a left-invariant f.a.p. measure on $\mathcal{P}(X)$.*

Proof. Choose a point $x \in X$ and define $\nu : \mathcal{P}(X) \to [0,1]$ by $\nu(A) = \mu(\{g \in G\,;\ gx \in A\})$. □

A consequence of this is the following

Proposition 4.13. *If an amenable group G acts on a space X then X cannot be G–paradoxical.*

Remark 4.14. This and the fact that the sphere $\mathbb{S}^2$ is $O(3)$–paradoxical implies that $O(3)$ is not amenable.

There exists a much stronger version of this:

Theorem 4.15 (Tarski's Theorem [Wag85], Corollary 9.2)**.** *Let G be a group acting on a space X and let E be a subset in X. Then E is not G–paradoxical if and only if there exists a G–left-invariant finitely additive measure $\mu : \mathcal{P}(X) \to [0,\infty]$ such that $\mu(E) = 1$.*

Remark 4.16. Theorem 4.15 and the Banach-Tarski paradox prove that there exists no $\mathrm{Isom}(\mathbb{R}^3)$–left-invariant finitely additive measure $\mu : \mathcal{P}(\mathbb{R}^3) \to [0,\infty]$ such that the measure of the unit ball is 1.

Proposition 4.17. *1. A subgroup of an amenable group is amenable.*

2. Let N be a normal subgroup of G. The group G is amenable if and only if N and G/N are amenable.

3. *The direct union of a directed system of amenable groups is amenable. That is: given a family of amenable groups $\{H_i\ ;\ i \in I\}$ with the property that for every $i, j \in I$ there exists $k \in I$ such that H_i, H_j are both subgroups of H_k, $G = \bigcup_{i\in I} H_i$ is amenable.*

Proof. (1) Let μ be a f.a.p. measure on an amenable group G, and let H be a subgroup. Let D be a subset of G intersecting each right coset Hg in exactly one point. Then $\nu(A) = \mu(AD)$ defines a left-invariant f.a.p. measure on H.

(2) "$\Rightarrow$" Assume that G is amenable and let μ be a f.a.p. measure on G. The subgroup N is amenable according to (1). For every subset $A \in G/N$ define $\nu(A) = \mu\left(\bigcup_{gN\in A} gN\right)$. One can easily check that it is a left-invariant f.a.p. measure on G/N. This is in fact a particular case of Lemma 4.12.

(2) "$\Leftarrow$" Let ν be a left-invariant f.a.p. measure on G/N, and λ a left-invariant f.a.p. measure on N. On every left coset gN one can define a f.a.p. measure by $\lambda_g(A) = \lambda(g^{-1}A)$. The H–left-invariance of λ implies that λ_g is independent of the representative g, i.e. $gN = g'N \Rightarrow \lambda_g = \lambda_{g'}$.

For every subset B in G define

$$\mu(B) = \int_{G/N} \lambda_g(B \cap gN) d\nu(gN)\,.$$

It is a G–left-invariant probability measure.

(3) The set $\{f : \mathcal{P}(G) \to [0,1]\ ;\ f \text{ function}\} = \prod_{\mathcal{P}(G)}[0,1]$ is compact according to Tychonoff's theorem (see Remark 3.16, 5).

Let ν_i be a left-invariant f.a.p. measure on H_i. For each $i \in I$ let $\mathcal{M}_i$ be the set of H_i–left-invariant f.a.p. measures on G. It is non-empty since it contains μ_i defined by $\mu_i(A) = \nu_i(A \cap H_i)$. Let us prove that $\mathcal{M}_i$ is closed. Let $f : \mathcal{P}(X) \to [0,1]$ be an element of $\prod_{\mathcal{P}(G)}[0,1]$ in the closure of $\mathcal{M}_i$. This implies that for every finite collection $A_1, ..., A_n$ of subsets of X and every $\epsilon > 0$ there exists μ in $\mathcal{M}_i$ such that $|f(A_j) - \mu(A_j)| \leq \epsilon$ for every $j \in \{1, 2, ..., n\}$. This implies that for every $\epsilon > 0$, $|f(X) - 1| \leq \epsilon$, $|f(A \sqcup B) - f(A) - f(B)| \leq 3\epsilon$ and $|f(gA) - f(A)| \leq 2\epsilon$, $\forall A, B \in \mathcal{P}(X)$ and $g \in H_i$. By letting $\epsilon \to 0$ we obtain that $f \in \mathcal{M}_i$.

Recall that if $\{V_i : i \in I\}$ is a family of closed subsets of a compact space X such that $\bigcap_{j\in J} V_j \neq \emptyset$ for any finite subset $J \subseteq I$ then $\bigcap_{i\in I} V_i \neq \emptyset$. (see [Dug66] or [Sut75]).

Consider a finite subset J of I. There exists $k \in I$ such that H_j is a subset of H_k for every $j \in J$. Then $\bigcap_{j\in J} \mathcal{M}_j$ contains $\mathcal{M}_k$, in particular it is non-empty. It follows from the above that $\bigcap_{i\in I} \mathcal{M}_i$ is non-empty. An element of it is clearly a f.a.p. measure, and it is G–left-invariant because $G = \bigcup_{i\in I} H_i$. □

Corollary 4.18. *Any group containing a subgroup which is free of rank two is non-amenable. In particular any non-Abelian free group is non-amenable.*

Proof. The first statement is due to Proposition 4.17, (1), and to Example 4.7. The second follows from the fact that any non-Abelian free group has a subgroup free of rank 2. □

Corollary 4.19. *A semidirect product $N \rtimes H$ is amenable if and only if N and H are amenable.*

Proof. The statement follows from (2). □

Corollary 4.20. *A group is amenable if and only if all its finitely generated subgroups are amenable.*

Proof. The direct part follows from (1). The converse part follows from (3), where, given the group G, I is the collection of all finite subsets in G, and for any $F \in I$, H_F is the subgroup of G generated by the elements in F. □

4.2 Equivalent definitions for finitely generated groups

Definition 4.21. A finitely generated group G is said to have *Følner's Property* if for every finite subset K of G and every $\epsilon > 0$ there exists a finite non-empty subset F such that for all $g \in K$

$$\frac{|gF \triangle F|}{|F|} \leq \epsilon \,. \tag{9}$$

Lemma 4.22. *Følner's Property is equivalent to the existence of a sequence of finite subsets (F_n) such that for every $g \in G$*

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0 \,. \tag{10}$$

A sequence (F_n) as above is called Følner sequence.

Proof. The converse implication is immediate. For the direct implication it suffices to note that, since G is countable, it can be written as $G = \{g_1, g_2, ..., g_n, ...\}$ and to apply Definition 4.21 to $K_n = \{g_1, g_2, ..., g_n\}$ and $\epsilon = \frac{1}{n}$ to obtain a set F_n. □

Remark 4.23. In both definitions of Følner's Property, one can take the action of G to the right, i.e. $\frac{|Fg \triangle F|}{|F|} \leq \epsilon$ in (9) etc. One way of formulating is equivalent to the other via inversion.

Exercise 4.24. Prove that the subsets $F_n = \mathbb{Z}^k \cap [-n, n]^k$ compose a Følner sequence for $\mathbb{Z}^k$.

Theorem 4.25. *Let G be a finitely generated group. The following are equivalent:*

1. *G is amenable;*
2. *G has Følner's Property.*

Proof of (2)⇒ (1). Let (F_n) be a Følner sequence, and let ω be a non-principal ultrafilter on $\mathbb{N}$. For every $A \subset G$ define

$$\mu(A) = \lim_{\omega} \frac{|A \cap F_n|}{|F_n|} \,.$$

It is easy to check that it is a f.a.p. measure. Now

$$\mu(gA) - \mu(A) = \lim_{\omega} \frac{|gA \cap F_n| - |A \cap F_n|}{|F_n|} = \lim_{\omega} \frac{|A \cap g^{-1}F_n| - |A \cap F_n|}{|F_n|} =$$

$$\lim_{\omega} \frac{|A \cap (g^{-1}F_n \setminus F_n)| - |A \cap (F_n \setminus g^{-1}F_n)|}{|F_n|} \leq \lim_{\omega} \frac{|A \cap (g^{-1}F_n \setminus F_n)| + |A \cap (F_n \setminus g^{-1}F_n)|}{|F_n|} =$$

$$\lim_{\omega} \frac{|A \cap (g^{-1}F_n \triangle F_n)|}{|F_n|} \leq \lim_{\omega} \frac{|g^{-1}F_n \triangle F_n|}{|F_n|} = 0 \,.$$

Similarly we prove that $\mu(A) - \mu(gA) \leq 0$, therefore $\mu(gA) = \mu(A)$.

Before proving the converse implication in Theorem 4.25 let us discuss some consequences of the implication already proven.

Proposition 4.26. *(1) If (F_n) is a Følner sequence in a finitely generated group G and ω is a non-principal ultrafilter on $\mathbb{N}$ then a left-invariant mean $m : \ell^\infty(G) \to \mathbb{C}$ may be defined by*

$$m(f) = \lim_\omega \frac{1}{|F_n|} \sum_{x \in F_n} f(x)$$

.

(2) For any $k \in \mathbb{N}$ the group $\mathbb{Z}^k$ is amenable and a mean $m : \ell^\infty\left(\mathbb{Z}^k\right) \to \mathbb{C}$ is defined by

$$m(f) = \lim_\omega \frac{1}{(2n+1)^k} \sum_{x \in \mathbb{Z}^k \cap [-n,n]^k} f(x)\,.$$

In particular for $k = 1$ a mean is

$$m(f) = \lim_\omega \frac{f(-n) + f(-n+1) + \cdots + f(n-1) + f(n)}{2n+1}\,.$$

Proof. (1) It suffices to note that $\mu(A) = m(\mathbf{1}_A)$ is the left invariant f.a.p. measure defined in the proof of (2)⇒ (1) above.

(2) is a consequence of (1) and Exercise 4.24. □

Proposition 4.27. *Every Abelian group is amenable.*

Proof. Indeed, every finitely generated Abelian group is isomorphic to a direct product of a finite Abelian group and $\mathbb{Z}^k$ for some $k \in \mathbb{N}$. Therefore by Examples 4.6 and 4.24 and Corollary 4.19 it is amenable. We finish the argument with Corollary 4.20. □

We shall prove the implication (1)⇒ (2) in Theorem 4.25 by proving the contrary, that is $\neg$ (2)⇒ $\neg$ (1). We shall in fact prove that a whole list of properties is equivalent to non-amenability. The only tool we need is a well-known theorem in graph theory.

Let $\mathcal{G}$ be a graph with set of vertices V and set of edges E. The graph is *locally finite* if the valency of every vertex is finite. For a set F of vertices we denote by $\partial_E F$ the set of vertices not in F but adjacent (i.e. connected by an edge) to a vertex in F.

A graph $\mathcal{G}$ is *bipartite* if the vertex set V splits as $V = Y \sqcup Z$ so that each edge $e \in E$ has one endpoint in Y and one endpoint in Z. We write $\mathcal{G} = Bip(Y, Z; E)$

Given two integers $k, l \geq 1$, a *perfect (k,l)–matching* of $Bip(Y, Z; E)$ is a subset M of the set of edges E such that each vertex in Y is endpoint of exactly k edges in M, while each vertex in Z is endpoint of exactly l edges in M.

Theorem 4.28 (Hall-Rado [Bol79], §III.2)**.** *Let $Bip(Y, Z; E)$ be a locally finite bipartite graph and let $k \geq 1$ be an integer such that:*

- *for any finite subset A in Y, $|\partial_E A| \geq k|A|$;*
- *for any finite subset B in Z, $|\partial_E B| \geq |B|$;*

Then $Bip(Y, Z; E)$ has a perfect $(k, 1)$–matching.

Given a discrete metric space (X, dist), two subsets Y, Z in X, and a real number $C \geq 0$, one can construct a bipartite graph $Bip_C(Y, Z)$ such that two vertices $y \in Y$ and $z \in Z$ are connected if and only if $\text{dist}(y, z) \leq K$.

For every metric space (X, dist) we denote by $\mathcal{B}(X)$ the set of maps $f : X \to X$ (not necessarily bijections) which are *bounded perturbations of the identity*, i.e. such that $\sup_{x\in X} \text{dist}(f(x), x)$ is finite.

Let (G, dist_S) be a finitely generated group endowed with a word metric. Let us investigate $\mathcal{B}(G)$. It contains all the right translations $R_g : G \to G$, $R_g(x) = xg$ (see Remark 2.22).

Lemma 4.29. *In a finitely generated group endowed with a word metric (G, dist_S), the set of maps $\mathcal{B}(G)$ is composed of* piecewise right translations*. That is, given a map $f \in \mathcal{B}(G)$ there exist finitely many elements $h_1, ..., h_n$ in G and a decomposition $G = T_1 \sqcup T_1 \sqcup ... \sqcup T_n$ such that f restricted to T_i coincides with R_{h_i}.*

Proof. Since $f \in \mathcal{B}(G)$ there exists a constant $R > 0$ such that for every $x \in G$, $\text{dist}(x, f(x)) \leq R$. This implies that $x^{-1}f(x) \in B(1, R)$. The ball $B(1, R)$ is a finite set. We enumerate its distinct elements $\{h_1, ..., h_n\}$. Thus for every $x \in G$ there exists h_i such that $f(x) = xh_i = R_{h_i}(x)$ for some $i \in \{1, 2, ..., n\}$. We define $T_i = \{x \in X \ ; \ f(x) = R_{h_i}(x)\}$. If there exists $x \in T_i \cap T_j$ then $f(x) = xh_i = xh_j$, which implies $h_i = h_j$, a contradiction. □

Theorem 4.30. *Let G be a finitely generated group. The following conditions are equivalent:*

(a) G is paradoxical;

(b) G does not have Følner's Property;

(c) G endowed with a word metric satisfies the doubling condition*: there exists a constant C such that for every finite non-empty subset F of G, $\mathcal{N}_C(F)$ has cardinality at least twice the cardinality of F;*

(d) when G is endowed with a word metric, there exists a constant $C > 0$ such that the graph $Bip_C(G, G)$ has a perfect $(2, 1)$–matching;

(e) there exists a map $f \in \mathcal{B}(G)$ such that for every $g \in G$ the pre-image $f^{-1}(g)$ contains exactly two elements.

*(f) (*Gromov's condition*) there exists a map $f \in \mathcal{B}(G)$ such that for every $g \in G$ the pre-image $f^{-1}(g)$ contains at least two elements.*

Proof. (a) ⇒ (b). If G would have Følner's property then by the proof of implication (2)⇒ (1) of Theorem 4.25 the group G would be amenable. This and Proposition 4.13 implies that G cannot be paradoxical.

(b) ⇒ (c). Assume that G does not satisfy Følner's property. Then there exists $\epsilon > 0$ and a finite set K such that for every finite subset F in G there exists $g \in K$ such that

$$\frac{|Fg \triangle F|}{|F|} > 2\epsilon \,.$$

We fix a word metric dist_S on G, and let C be the maximum of all $\text{dist}_S(1, g)$ with $g \in K$. Let F be an arbitrary finite set. According to the above there exists $g \in K$ such that either $|Fg \setminus F|$ or $|F \setminus Fg|$ is at least $\epsilon|F|$. Note that $|F \setminus Fg| = |Fg^{-1} \setminus F|$. Both $Fg \setminus F$ and $Fg^{-1} \setminus F$ are in $\mathcal{N}_{2C}(F) \setminus F$. Indeed for every $x \in F$, $\text{dist}(xg, x) = \text{dist}(1, g) \leq C$. Thus $|\mathcal{N}_{2C}(F)| \geq (1 + \epsilon)\,|F|$.

We have thus obtained that for every finite set F, $|\mathcal{N}_{2C}(F)| \geq (1+\epsilon)|F|$. An induction argument implies that for every finite set F, $|\mathcal{N}_{2Cn}(F)| \geq (1+\epsilon)^n |F|$. For n large enough we obtain $(1+\epsilon)^n \geq 2$.

(c) $\Rightarrow$ (d). The bipartite graph $Bip_C(G,G)$ for the constant C in (c) is locally finite. For any finite subset A in G, since $|\mathcal{N}_C(A)| \geq 2|A|$ it follows that $|\partial_E A| \geq 2|A|$ while clearly for any finite subset B in the second copy of G $|\partial_E B| \geq |B|$. It follows by Theorem 4.28 that $Bip_C(G,G)$ has a perfect $(2,1)$–matching.

(d) $\Rightarrow$ (e). The matching in (d) defines a map as in (e).

(e) $\Rightarrow$ (a). Let $f : G \to G$ be such that $|f^{-1}(y)| = 2$ for every $y \in G$ and $f \in \mathcal{B}(X)$. The latter property implies by Lemma 4.29 that there exists a finite set $\{h_1, ..., h_n\}$ and a decomposition $G = T_1 \sqcup ... \sqcup T_n$ such that f restricted to T_i coincides with R_{h_i}.

For every $y \in G$ we have that $f^{-1}(y)$ is composed of two elements which we order as $\{y_1, y_2\}$. This gives a decomposition of G into $Y_1 \sqcup Y_2$. Now we decompose $Y_1 = A_1 \sqcup ... \sqcup A_n$, where $A_i = Y_1 \cap T_i$, and likewise $Y_2 = B_1 \sqcup ... \sqcup B_n$, where $B_i = Y_2 \cap T_i$. Clearly $A_1 h_1 \sqcup ... \sqcup A_n h_n = G$ and $B_1 h_1 \sqcup ... \sqcup B_n h_n = G$.

Thus we proved that statements (a)-(e) are equivalent.

(e) $\Rightarrow$ (f) is obvious. We prove (f) $\Rightarrow$ (c). Indeed let $f \in \mathcal{B}(G)$ be such that $|f^{-1}(g)| \geq 2$. Let $R > 0$ be such that for every $x \in G$, $\mathrm{dist}(x, f(x)) < R$. Then for every finite set F, $\mathcal{N}_R(F)$ contains $\bigsqcup_{y \in F} f^{-1}(y)$, whence $|\mathcal{N}_R(F)| \geq 2|F|$. □

Theorem 4.30 gives that if G does not have Følner's property then G is paradoxical hence non-amenable. This completes the proof of Theorem 4.25. Another consequence is

Corollary 4.31. *A finitely generated group is either paradoxical or amenable.*

This is a weaker version of Tarski's Alternative Theorem 4.15.

We can also provide a third proof that the free group on two generators is non-amenable: consider the map $f : F_2 \to F_2$ which consists in deleting the last letter in every reduced word. This satisfies Gromov's condition.

We are now able to relate amenable groups to the Banach–Tarski paradox.

Proposition 4.32. *1. The group of isometries* $\mathrm{Isom}(\mathbb{R}^n)$ *with* $n = 1, 2$ *is amenable.*

2. The group of isometries $\mathrm{Isom}(\mathbb{R}^n)$ *with* $n \geq 3$ *is non-amenable.*

Proof. (1) This follows from Corollary 4.19 applied to $\mathrm{Isom}(\mathbb{R}^n)$, with $H = O(n)$ and $N = \mathbb{R}^n$. Note that $O(n)$ itself is amenable due to the fact that for $n = 1$ it is isomorphic to $\mathbb{Z}_2$, while for $n = 2$ it is a semidirect product of $SO(2)$ which is the group of rotations, hence Abelian, and a copy of $\mathbb{Z}_2$ generated by the element with diagonal $(1, -1)$.

(2) This follows from Proposition 4.17, (1), and from Example 2.8. □

The observation that the existence of a free subgroup excludes amenability was first made by J. von Neumann in [vN28], the very paper in which he introduced the notion of amenable group, under the name of measurable group. It is this observation that raised the following question:

Question 4.33 (the von Neumann problem)**.** *Does every non-amenable group contain a free non-Abelian subgroup ?*

The question is implicit in [vN29], and it was formulated explicitly by Day [Day57, §4].

When restricted to the class of linear groups (i.e. of groups which have faithful finite-dimensional linear representations), Question 4.33 has an affirmative answer, moreover a linear group without any free non-Abelian subgroup is virtually solvable.

Theorem 4.34 (Tits' alternative [Tit72])**.** *Let G be a subgroup of $GL(n, \mathbb{K})$ for some integer $n \geq 1$ and some field $\mathbb{K}$ of characteristic zero. Then either G has a non-Abelian free subgroup or G is virtually solvable.*

Remark 4.35. Other classes satisfying the Tits' alternative are:

1. finitely generated subgroups of $GL(n, \mathbb{K})$ for some integer $n \geq 1$ and some field $\mathbb{K}$ of finite characteristic [Tit72];

2. subgroups of Gromov hyperbolic groups ([Gro87, §8.2.F], [GdlH90, Chapter 8]).

The first examples of non-amenable groups with no (non-Abelian) free subgroups were given in [Ol'80]. In [Ady82] it was shown that the free Burnside groups $B(n, m)$ with $n \geq 2$ and $m \geq 665$, m odd, are also non-amenable. The first finitely presented examples of non-amenable groups with no (non-Abelian) free subgroups were given in [OS02].

M. Day has defined the class of *elementary amenable groups* as the smallest class of groups which contains finite and Abelian groups, and which is closed under the operations described in Proposition 4.17. He asked in [Day57] the following question:

Question 4.36. *Is every amenable group elementary amenable ?*

The answer is again negative: this was first shown by Grigorchuk ([Gri84], [Gri85]), then a more recent elegant argument was provided by Stepin [Ste96].

4.3 Amenability and growth

For some finitely generated groups a Følner sequence may be entirely composed of balls. This is already quite apparent for Abelian groups. We shall see other classes of groups with this property in the sequel.

The *growth function* of a finitely generated group G endowed with the word length dist_S corresponding to a finite generating set S is the function $\mathfrak{G}_S : \mathbb{N} \to \mathbb{N}$, $\mathfrak{G}_S(n) = |B_S(1, n)|$.

Notation 4.37. *For two functions $f, g : X \to \mathbb{R}$ with $X \subset \mathbb{R}$ we write $f \ll g$ if there exist $a, b, c, d > 0$ such that $f(x) \leq ag(bx + c) + d$ for every $x \in \mathbb{R}$. If $f \ll g$ and $g \ll f$ then we write $f \asymp g$.*

If S, S' are two finite generating functions of G then $\mathfrak{G}_S \asymp \mathfrak{G}_{S'}$.

Examples 4.38.

1. If $G = Z^k$ and S is the standard generating set then $\mathfrak{G}_S \asymp x^k$.

2. If $G = F_k$ is the free group of finite rank k and S is the set of $2k$ generators then $\mathfrak{G}_S(n) = 2k(2k - 1)^{n-1}$.

Exercise 4.39. Prove the following properties of the growth function:

1. If G is infinite, $\mathfrak{G}_S$ is strictly increasing.

2. The growth function is sub-multiplicative:

$$\mathfrak{G}_S(n + m) \leq \mathfrak{G}_S(n)\mathfrak{G}_S(m)\,.$$

3. If $|S| = k$ then $\mathfrak{G}_S(n) \leq 2k(2k-1)^{n-1}$, where we recognise in the latter the growth function of the free group of rank k.

Property (2) implies that the following limit is well-defined:

$$\gamma_S = \lim_{n\to\infty} \mathfrak{G}(n)^{\frac{1}{n}}.$$

Property (1) implies that $\mathfrak{G}_S(n) \geq n$, whence $\gamma_S \geq 1$. If $\gamma_S > 1$ then G is said to be of *exponential growth.* If $\gamma_S = 1$ then G is said to be of *sub-exponential growth.*

Proposition 4.40. *A finitely generated group of sub-exponential growth is amenable with a Følner sequence composed of balls of centre* 1.

Proof. Assume there exists $\epsilon > 0$ and N such that for any $n \geq N$, $\sup_{s\in S} |sB(1,n) \triangle B(1,n)| \geq \epsilon|B(1,n)|$. Let n be an arbitrary integer larger than N and let s be the generator for which the supremum above is attained. Note that $sB(1,n) = B(s,n)$. As $|B(s,n) \triangle B(1,n)| = |B(s,n) \setminus B(1,n)| + |B(1,n) \setminus B(s,n)|$, either $|B(s,n) \setminus B(1,n)|$ or $|B(1,n) \setminus B(s,n)|$ is at least $\frac{\epsilon}{2}|B(1,n)|$. Assume that the former case occurs. As $\mathrm{dist}_S(1,s) = 1$ we have that $B(s,n) \subset B(1,n+1)$. Then $|B(1,n+1)| \geq |B(1,n)| + |B(s,n) \setminus B(1,n)| \geq \left(1+\frac{\epsilon}{2}\right)|B(1,n)|$. If $|B(1,n) \setminus B(s,n)| \geq \frac{\epsilon}{2}|B(1,n)|$ then $|B(1,n+1)| \geq |B(s,n)| + |B(1,n) \setminus B(s,n)| \geq \left(1+\frac{\epsilon}{2}\right)|B(1,n)|$. The previous inequality and an induction argument then implies that for any $n \geq N$, $|B(1,n)| \geq \left(1+\frac{\epsilon}{2}\right)^{n-N}|B(1,N)|$. In particular the growth is exponential, a contradiction.

It follows that for every $\epsilon > 0$ and N there exists $n \geq N$ such that $\sup_{s\in S} |sB(1,n) \triangle B(1,n)| \leq \epsilon|B(1,n)|$. This allows to find an increasing sequence of integers (n_k) such that $\sup_{s\in S} |sB(1,n_k) \triangle B(1,n_k)| \leq \frac{1}{k}|B(1,n_k)|$. The sequence $(B(1,n_k))_{k\in\mathbb{N}}$ is therefore a Følner sequence. □

Conjecture 4.41 (J. Milnor [Mil68b]). *The growth of a finitely generated group is either polynomial (i.e. $\mathfrak{G} \ll x^d$) or exponential (i.e. $\mathfrak{G} \gg a^x$ for some $a > 1$).*

The conjecture is true for linear groups by Theorem 4.34 and still open for finitely presented groups, but it is disproved for finitely generated groups: there exist subgroups of intermediate growth, examples due to Grigorchuk.

5 Abelian, nilpotent and solvable groups.

We shall follow [KL95], [dDW84] and [Gro81].

5.1 Definitions, examples, amenability

Let G be a group. The *commutator of two elements* h, k is

$$[h,k] = hkh^{-1}k^{-1}.$$

Note that $[h,k]^{-1} = [k,h]$. Also $hk = [h,k]kh$. Thus, two elements h, k commute if and only if $[h,k] = 1$.

Let H, K be two subgroups of G. We denote by $[H,K]$ the subgroup generated by all commutators $[h,k]$ with $h \in H$, $k \in K$.

The *commutator subgroup* $G' = [G,G]$ is the subgroup generated by all commutators in G. The group G is Abelian if and only if $G' = \{1\}$. The *abelianization* of a group G is the quotient $G_{ab} = G/G'$.

Exercise 5.1. Let $\varphi : G \to A$ be a homomorphism with A Abelian subgroup. Prove that φ factors through the abelianization, i.e. given $p : G \to G_{ab}$ the canonical projection, there exists a homomorphism $\overline{\varphi} : G_{ab} \to A$ such that $\varphi = \overline{\varphi} \circ p$.

We define by induction the *iterated commutator subgroup* $G^{(k)}$. We define: $G^{(0)} = G$, $G^{(1)} = G'$, $G^{(k+1)} = \left(G^{(k)}\right)'$.

A *descending series of normal subgroups* is a series

$$G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_n \supseteq \cdots$$

such that N_{i+1} is a normal subgroup in N_i for every $i \geq 0$.

The descending series of normal subgroups $G \trianglerighteq G' \trianglerighteq \cdots \trianglerighteq G^{(k)} \trianglerighteq G^{(k+1)} \trianglerighteq \dots$ is called the *derived series* of the group G.

Definition 5.2. A group G is *solvable* if there exists k such that $G^{(k)} = \{1\}$.

Exercise 5.3. Prove that if N is a normal subgroup in G and both N and G/N are solvable then G is solvable.

Example 5.4. From the above we may deduce that $Isom(\mathbb{R}^n)$ for $n = 1, 2$, is solvable.

Example 5.5. The group $\mathcal{T}_n$ of invertible upper triangular $n \times n$ matrices is solvable.

Proof. Let $\mathcal{T}_{n,k}$ be the subgroup of $\mathcal{T}_n$ composed of matrices (a_{ij}) such that $a_{ij} = \delta_{ij}$ for $j < i+k$.

It can be easily proved that $\mathcal{T}_n' \subset \mathcal{T}_{n,1}$. Next we prove that $(\mathcal{T}_{n,k})' \subset \mathcal{T}_{n,k+1}$. Indeed the map

$$\begin{array}{ccc} \mathcal{T}_{n,k} & \to & \mathbb{R}^{n-k} \\ A = (a_{ij}) & \mapsto & (a_{1,k+1}, a_{2,k+2}, \dots, a_{n-k,n}) \end{array}$$

is a homomorphism with Abelian image, therefore the kernel, which coincides with $\mathcal{T}_{n,k+1}$, contains $(\mathcal{T}_{n,k})'$. □

Proposition 5.6. *Every solvable group is amenable.*

Proof. We argue by induction on the minimal k such that $G^{(k)} = \{1\}$. If $k = 1$ then G is Abelian and the statement is true. Assume that it is true for k and take G such that $G^{(k+1)} = \{1\}$ and $G^{(i)} \neq \{1\}$ for any $i \leq k$. Then $G^{(k)}$ is Abelian and $\bar{G} = G/G^{(k)}$ has the property that $\bar{G}^{(k)} = \{1\}$, whence by the inductive hypothesis $\bar{G}$ is amenable. This and Proposition 4.17, (2), imply that G is amenable. □

We define by induction another series of normal subgroups of G:

$$C^0 G = G\,,\ C^{n+1} G = [G, C^n G]\,.$$

The descending series of normal subgroups $G \trianglerighteq C^1 G \trianglerighteq \cdots \trianglerighteq C^n G \trianglerighteq C^{n+1} G \trianglerighteq \dots$ is called the *lower central series* of the group G.

Definition 5.7. A group G is called (k-*step*) *nilpotent* if there exists k such that $C^k G = \{1\}$. The minimal such k is the *class* of G.

Examples 5.8.
1. An Abelian group is nilpotent of class 0.
2. The group of upper triangular $n \times n$ matrices with 1 on the diagonal is nilpotent of class $n-1$.

3. The *Heisenberg group*

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ 0 & 1 & 0 & \dots & y_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & y_1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} ; \, x_1, ..., x_n, y_1, ..., y_n, z \in \mathbb{R} \right\}$$

is nilpotent of class 2.

If we take $x_1, ..., x_n, y_1, ..., y_n, z \in \mathbb{Z}$ we obtain a subgroup of the first, the *discrete Heisenberg group*. The latter is finitely generated, with generators the elementary matrices E_{ij} with $(i, j) \in \{(1, 2), \dots, (1, n-1), (2, n), \dots, (n-1, n)\}$.

Both groups are nilpotent of class 2. Indeed $C^1 H_{2n+1}$ is the subgroup with all $x_i = 0$ and $y_i = 0$.

Remark 5.9. 1. The commutator subgroup $G' = C^1 G$ of a nilpotent group is finitely generated [Hal76, Section §10.2].

2. The same does not hold for solvable groups.

5.2 Growth of nilpotent groups

Lemma 5.10. *Let $G = \langle S \rangle$ be a finitely generated nilpotent group, and let $G' = \langle X \rangle$ be its commutator subgroup, also finitely generated (see Remark 5.9). Then for every $h \in G'$, $\mathrm{dist}_X(1, h) \preceq [\mathrm{dist}_S(1, h)]^{n+1}$ where $n \in \mathbb{N}$ is the class of G.*

Proof. Up to replacing G with a finite index subgroup it may be assumed that $G/G' \simeq \mathbb{Z}^m$. We choose $S = S_0 \cup S_1 \cup ... \cup S_n$, where n is the class of G, and $S_i^{-1} = S_i$, such that S_0 projects onto a generating set of $\mathbb{Z}^m$, $S_i \in C^i G$, and S_{i+1} contains all $[x, h_i]$ with $x \in S_0$ and $h_i \in S_i$.

Given a word w in S we denote by $|w|_S$ its length and by $|w|_{S_i}$ the number of letters from S_i appearing in w.

We move every letter $x \in S_0$ appearing in w such that it appears as first letter. Each "crossing" $yx \mapsto xy[y^{-1}, x^{-1}]$ with $y \in S_i$ results in the introduction of a commutator, which is a letter in S_{i+1}. We thus obtain a new word w_1 with $|w_1|_{S_{i+1}} \leq |w|_{S_{i+1}} + |w|_{S_i}$.

We move thus all letters in S_0 in the beginning and obtain a sequence of words $w_0 = w, w_1, ..., w_m = x_1 ... x_m u$ with $x_i \in S_0$ and u a word in $S_1 \cup ... \cup S_n$. All $|w_i|_{S_0}$ are the same, while $|w_{k+1}|_{S_{i+1}} \leq |w_k|_{S_{i+1}} + |w_k|_{S_i}$.

We thus obtain

$$|w_m|_{S_{i+1}} \leq |w|_{S_{i+1}} + m|w|_{S_i} + m(m-1)|w|_{S_{i-1}} + \dots + \frac{m!}{(m-i-1)!}|w|_{S_0} \leq$$

$$m^{i+1} \sum_{k=0}^{i+1} |w|_{S_j} \leq m^{i+1} |w|_S \leq |w|_S^{i+2} .$$

It follows that $|u|_S = \sum_{i=1}^n |u|_{S_i} = \sum_{i=1}^n |w_m|_{S_i} \leq \sum_{i=1}^n |w|_S^{i+1} \leq n|w|_S^{n+1}$.

If the word w is the shortest word in S representing an element g from G' then $w_m = u$, and for $X = S_1 \cup ... \cup S_n$ $|g|_X \leq |u|_S \leq n|w|_S^{n+1}$. □

Proposition 5.11. *Every nilpotent group has at most polynomial growth.*

Proof. We argue by induction on the class. For $n = 1$ the group is Abelian and the statement is obviously true. Assume it is true for n, and let G be a nilpotent group of class $n+1$. Then G' is of class n hence of polynomial growth, $\mathfrak{G}_{G'}(x) \preceq x^d$. We choose a generating set $S = S_0 \cup S_1$ as above. Then every $g \in B(1, R)$ can be written as $g_0 g_1$ with g_0 a word in S_0 of length at most R, and $g_1 \in G'$, $|g_1|_S$ at most $2R$. Then $|g_1|_{S_1} \leq CR^{n+1}$. We thus obtain that $\mathfrak{G}_S(G) \leq KR^m R^{(n+1)d}$. □

A more precise estimate can be found for the growth of a nilpotent group:

Theorem 5.12. *([Bas72]) If G is nilpotent and each $C^i G / C^{i+1} G \simeq A_i^f \times \mathbb{Z}^{m_i}$ then the growth function $\mathfrak{G}_G$ of G satisfies $\mathfrak{G}_G(x) \asymp x^k$ where $k = \sum_i (i+1) m_i$.*

5.3 Growth of solvable groups

We begin by discussing automorphisms of Abelian groups.

Lemma 5.13. *If a matrix M in $GL(n, \mathbb{Z})$ has all eigenvalues equal to 1 then there exists a nested sequence*

$$\{1\} \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_{n-1} \subseteq \mathbb{Z}^n$$

such that $\Lambda_i \simeq \mathbb{Z}^i$, $\Lambda_{i+1}/\Lambda_i \simeq \mathbb{Z}$ and $M(\Lambda_i) = \Lambda_i$, and M acts on Λ_{i+1}/Λ_i as identity.

Proof. Since M has eigenvalue 1 there exists a vector $v = (v_1, .., v_n) \in \mathbb{Z}^n$ such that $\mathrm{hcf}(v_1, .., v_n) = 1$ and $Mv = v$. Then M induces an automorphisms of $\mathbb{Z}^n / \mathbb{Z}v \simeq \mathbb{Z}^{n-1}$ and the matrix of this automorphism has only 1 as eigenvalue. Thus there exists $w + \mathbb{Z}v$ such that $M(w + \mathbb{Z}v) = w + \mathbb{Z}v$.

An inductive argument yields the conclusion. □

Lemma 5.14. *If a matrix M in $GL(n, \mathbb{Z})$ has all eigenvalues of absolute value 1 then all eigenvalues are roots of unity.*

Proof. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of this matrix listed with multiplicity. Then for every $k \in \mathbb{Z}$,

$$\mathrm{tr} M^k = \sum_{i=1}^{n} \lambda_i^k .$$

Now $v_k = (\lambda_1^k, ..., \lambda_n^k)$ is a vector in $(\mathbb{S}^1)^n$, which is compact, therefore a subsequence v_{k_l} converges to a vector v, hence the subsequence $v_{m_l} = v_{k_{l+1}} v_{k_l}^{-1}$ converges to $(1, ..., 1)$. It follows that $\mathrm{tr} M^{m_l} \to n$, on the other hand $\mathrm{tr} M^{m_l} \in \mathbb{Z}$, hence for l large enough $\mathrm{tr} M^{m_l} = n$. In particular $\sum_{i=1}^n \mathrm{Re}\, \lambda_i^{m_l} = n$ and since all terms in the sum are in $[-1, 1]$ it follows that they are all 1. □

Lemma 5.15. *If a matrix M in $GL(n, \mathbb{Z})$ has one eigenvalue λ of absolute value at least 2 then there exists a vector $v \in \mathbb{Z}^n$ such that the following map is injective:*

$$\bigoplus_{n \in \mathbb{Z}, n \geq 0} \mathbb{Z}_2 \to \mathbb{Z}^n ,$$

$$(s_n)_n \mapsto s_0 v + s_1 M v + ... + s_n M^n v + ...$$

Proof. The matrix M defines an automorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$, $\varphi(v) = Mv$. The dual map φ^* has matrix M^T in the dual canonical basis. Therefore it has the eigenvalue λ. It follows that there exists a linear form $f : \mathbb{C}^n \to \mathbb{C}$ such that $\varphi^*(f) = f \circ \varphi = \lambda f$.

Take $v \in \mathbb{Z}^n \setminus \ker f$. Assume that the considered map is not injective. It follows that for some $(s_n)_n$ not constant 0 we have that $s_0 v + s_1 Mv + ... + s_n M^n v + ... = 0$. Let N be the largest integer such that $s_N = 1$. Then $s_0 v + s_1 Mv + ... + s_{N-1} M^{N-1} v = -M^N v$. By applying f to the equality we obtain $\left(s_0 + s_1\lambda + \cdots + s_{N-1}\lambda^{N-1}\right) f(v) = -\lambda^N f(v)$, whence

$$|\lambda|^N \leq \sum_{i=1}^{N-1} |\lambda|^i = \frac{|\lambda|^N - 1}{|\lambda| - 1} \leq |\lambda|^N - 1\,,$$

a contradiction. □

Proposition 5.16. *Let G be a finitely generated nilpotent group and let $\varphi : G \to G$ be an automorphisms of it. Then the semidirect product $S = G \rtimes_\varphi \mathbb{Z}$ (which is solvable) is*

1. *either virtually nilpotent;*
2. *or with the growth function satisfying $\mathfrak{G}_S((n+1)^2) \geq 2^n$ for every $n \in \mathbb{N}$.*

Proof. Note that $G \rtimes_{\varphi^N} \mathbb{Z}$ is a subgroup of finite index in $G \rtimes_\varphi \mathbb{Z}$. Thus we may replace φ by some power of it, and take a finite index subgroup in our arguments.

The automorphism φ preserves the lower central series hence it induces homomorphisms φ_i of each of the Abelian groups $A_i = C^iG/C^{i+1}G$. Each A_i is isomorphic to $A_i^f \times \mathbb{Z}^{m_i}$ for some finite Abelian group A_i^f and some $m_i \in \mathbb{Z}, m_i \geq 0$. Therefore φ_i induces an automorphism of $\mathbb{Z}^{m_i}$, thus it defines a matrix M_i in $GL(m_i, \mathbb{Z})$. We have two cases.

(1) All M_i only have eigenvalues of absolute value 1, hence roots of unity. Then for some large enough power φ^N, the matrices M_i have all eigenvalues 1, hence we may assume from the beginning that all matrices M_i have all eigenvalues 1. This and Lemma 5.13 implies that the lower central series is a subsequence of a series

$$\{1\} = H_n \subseteq H_{n-1} \subseteq \cdots \subseteq H_1 \subseteq H_0 = G$$

such that H_i/H_{i+1} is cyclic, φ preserves each H_i and induces on H_i/H_{i+1} the identity action. Let t denote the generator of the factor $\mathbb{Z}$. We have that for every $g \in G$, $tgt^{-1} = \varphi(g)$. This and the above implies that $t^k(h_iH_{i+1})t^{-k} = h_iH_{i+1}$, that is $[t^k, h_i] \in H_{i+1}$.

We have an exact sequence $1 \to G \to S \to \mathbb{Z} \to 1$, and the projection of $C^1S = [S, S]$ onto $\mathbb{Z}$ is 1, hence $C^1S \leq G$. To end the argument it suffices to prove that $[S, H_i] \subseteq H_{i+1}$ for every $i \geq 0$.

For every $h_i \in H_i$ and $gt^k \in S$, $[h_i, gt^k] = h_igt^kh_i^{-1}t^{-k}g^{-1} = h_igh_i^{-1}[h_i, t^k]g^{-1} = [h_i, g]gh_{i+1}g^{-1} \in H_{i+1}$. This proves that $[S, H_i] \subseteq H_{i+1}$ for every $i \geq 0$, which implies that S is nilpotent, since G is.

(2) Assume that some M_i has an eigenvalue with absolute value strictly greater than 1. Up to replacing φ with φ^N we may assume that it has an eigenvalue with absolute value at least 2.

Lemma 5.15 applied to $\mathbb{Z}^{m_i}$ (where $C^iG/C^{i+1}G \simeq A_i^f \times \mathbb{Z}^{m_i}$) and to M_i implies that there exists an element $g \in C^iG$ which projects onto a vector $v \in \mathbb{Z}^{m_i}$ such that distinct elements $(s_n) \in \bigoplus_{n\geq 0} \mathbb{Z}_2$ define distinct elements $s_0v + s_1M_iv + ... + s_nM_i^nv + ...$ in $\mathbb{Z}^{m_i}$. Then the elements $g^{s_0}(tgt^{-1})^{s_1}...(t^ngt^{-n})^{s_n}...$ are also pairwise distinct. They define 2^n elements of length at most $(n+1)^2$. □

We continue with a digression on semidirect products and short exact sequences.

Lemma 5.17. *A short exact sequence* $1 \to N \to G \to Q \to 1$ *corresponds to a decomposition of* G *as a semidirect product (i.e. there exists a subgroup* H *such that* $G = N \rtimes H$*) if and only if there exists a subgroup* H *in* G *such that the projection* $\pi : G \to Q$ *restricted to* H *becomes an isomorphism.*

Proof. The direct implication is obvious.

Assume that there exists H such that $\pi|_H$ is an isomorphism. The fact that it is surjective implies that $G = NH$. The fact that it is injective implies that $H \cap N = \{1\}$. □

Remark 5.18. It is not always the case that the quotient in a short exact sequence corresponds to a subgroup in the middle group. This can be seen in the short exact sequence corresponding to the projection of a free group of rank k onto its abelianization $\mathbb{Z}^k$. Still in some cases this is true.

Lemma 5.19. *A short exact sequence* $1 \to N \to G \to \mathbb{Z} \to 1$ *corresponds to a decomposition of* G *as a semidirect product* $G = N \rtimes H$*, where* $H \simeq \mathbb{Z}$*.*

Proof. Indeed consider an element $t \in G$ projecting onto the generator 1 of $\mathbb{Z}$, and take $H = \langle t \rangle$. Clearly t has infinite order, hence $H \simeq \mathbb{Z}$. Also the projection of H onto $\mathbb{Z}$ is surjective and injective. □

Proposition 5.20. *Consider a short exact sequence* $1 \to N \to G \to \mathbb{Z} \to 1$*. If* G *has polynomial growth then* N *is finitely generated. Moreover if the growth function of* G *satisfies* $\mathfrak{G}_G(x) \preceq x^d$ *then the growth function of* N *satisfies* $\mathfrak{G}_N(x) \preceq x^{d-1}$*.*

Proof. According to Lemma 5.19 there exists $t \in G$ such that $G = N \rtimes \langle t \rangle$. Let $\{s_1, ..., s_m\}$ be a generating set in G. Then $s_i = n_i t^{k_i}$ for $k_i \in \mathbb{Z}$ and $i \in \{1, 2, ..., n\}$. It follows that $\{n_1, ..., n_m, t\}$ is also a generating set of G. Note that in every word $w(n_1, ..., n_m, t)$ representing an element in N the exponents of t must add up to 0. It follows that N is generated by the infinite set $\{t^i n_j t^{-i} \mid i \in \mathbb{Z}, j \in \{1, ..., m\}\}$. We denote $t^i n_j t^{-i}$ by g_{ij}. For every $j \in \{1, ..., m\}$ the products

$$g_{0j}^{\alpha_0} g_{1j}^{\alpha_1} ... g_{kj}^{\alpha_k} \,, \quad \text{with } \alpha_i \in \{0, 1\}$$

are 2^k words of length at most $(k+1)^2$. Since G has polynomial growth it follows that two distinct such words are equal in G, that is for some $l \le k$,

$$g_{0j}^{\alpha_0} g_{1j}^{\alpha_1} ... g_{lj}^{\alpha_l} = g_{0j}^{\beta_0} g_{1j}^{\beta_1} ... g_{lj}^{\beta_l} \,, \quad \text{and } \alpha_l \neq \beta_l \,.$$

Then $g_{lj} \in \langle g_{0j}, g_{1j}, ..., g_{(l-1)j} \rangle$. It follows that $g_{(l+1)j} = t g_{lj} t^{-1} \in t \langle g_{0j}, g_{1j}^{\alpha_1}, ..., g_{(l-1)j} \rangle t^{-1} = \langle g_{1j}, g_{2j}^{\alpha_1}, ..., g_{lj} \rangle \le \langle g_{0j}, g_{1j}^{\alpha_1}, ..., g_{(l-1)j} \rangle$. An induction implies that for all $i \ge l$,

$$g_{ij} \in \langle g_{0j}, g_{1j}^{\alpha_1}, ..., g_{(l-1)j} \rangle \,.$$

The same argument can be done for $i \le 0$. We thus obtain that N is generated by a finite set $\{g_{ij} \mid i \in \{-l, ..., l\},\ j \in \{1, ..., m\}\}$.

Now consider X a finite generating set for N and $X \cup \{t\}$ the generating set for G. For an arbitrary $n \ge 1$ let $\mathfrak{G}_N(n) = M$, where the growth function on N is taken with respect to the word metric defined by X. Then $B_H(1, n)$ contains M distinct elements $\{n_1, ..., n_M\}$. The elements in the following set are also pairwise distinct

$$\{n_i t^j \mid i \in \{1, 2, ..., M\},\ -n \le j \le n\}$$

since right cosets Nt^j are pairwise disjoint. They are contained in $B_G(1, 2n)$. It follows that $(2n+1)M \leq C(2n)^d$. We thus obtain that $\mathfrak{G}_N(n) \preceq n^{d-1}$.

□

Corollary 5.21. *A solvable group has polynomial growth if and only if it is virtually nilpotent.*

Proof. The converse implication was proved in Section 5.2. Let G be a solvable group such that $\mathfrak{G}_G(x) \preceq x^d$. We argue by induction on d. The case $d = 0$ corresponds to solvable groups that are finite. Assume that the statement is true for d and let G be a solvable group with $\mathfrak{G}_G(x) \preceq x^{d+1}$. Possibly by taking a finite index subgroup we may assume that $G_{ab} \simeq \mathbb{Z}^m$ and this allows us to construct an onto homomorphism $G \to \mathbb{Z}$, hence a short exact sequence $1 \to N \to G \to \mathbb{Z} \to 1$. By Proposition 5.20, N is finitely generated and $\mathfrak{G}_N(x) \preceq x^d$. The induction hypothesis implies that N is virtually nilpotent. Proposition 5.16 allows to finish the argument. □

Proposition 5.16 and Corollary 5.21 are particular cases of the following dichotomy due to J. A. Wolf and to J. Milnor.

Theorem 5.22 ([Wol68], [Mil68a]). *A finitely generated solvable group either has polynomial growth and is virtually nilpotent or has exponential growth.*

6 Ultralimits, asymptotic cones, examples

6.1 Definition, preliminaries

The notion of asymptotic cone was defined in an informal way in [Gro81], and then rigorously in [dDW84] and [Gro93]. The idea is to construct, for a given metric space, an image of it seen from infinitely far away. The main tool in this construction is a non-principal ultrafilter ω on $\mathbb{N}$.

The power of asymptotic cones lies in the fact that they capture both geometric and logical properties of infinite groups.

Convention 6.1. Since all ultrafilters are from now on non-principal and on $\mathbb{N}$, we omit mentioning both properties henceforth.

One has to take a sequence of positive numbers d_n diverging to infinity, and try to construct a limit of the sequence of metric spaces $\left(X, \frac{1}{d_n}\mathrm{dist}\right)$.

As in the formal construction of completion, one can simply take the set $\mathcal{S}(X)$ of all sequences (x_n) in X and try to define a metric on this space by

$$\mathrm{dist}_\omega(x, y) = \lim_\omega \frac{\mathrm{dist}(x_n, y_n)}{d_n}, \ \text{ for } x = (x_n),\, y = (y_n)\,.$$

The problem is that the latter limit can be $+\infty$, or it can be zero for two distinct sequences.

To avoid the situation $\mathrm{dist}_\omega(x, y) = +\infty$, one restricts to a subset of sequences defined as follows. For a fixed sequence $e = (e_n)$, consider

$$\mathcal{S}_e(X) = \left\{ (x_n) \in X^{\mathbb{N}}\,;\ \left(\frac{\mathrm{dist}(x_n, e_n)}{d_n}\right) \text{ is a bounded sequence} \right\}. \tag{11}$$

To deal with the situation when $\mathrm{dist}_\omega(x, y) = 0$ while $x \neq y$, one uses the classical trick of taking the quotient for the equivalence relation

$$x \sim y \Leftrightarrow \mathrm{dist}_\omega(x, y) = 0\,.$$

The quotient space $\mathcal{S}_e(X)/\sim$ is denoted $\mathrm{Con}_\omega(X;e,d)$ and it is called *the asymptotic cone of X with respect to the ultrafilter ω, the scaling sequence $d=(d_n)$ and the sequence of observation centers e.*

A sequence of subsets (A_n) in X gives rise to a *limit subset* in the cone, defined by

$$\lim_\omega(A_n)=\{\lim_\omega(a_n)\mid a_n\in A_n,\,\forall\, n\in\mathbb{N}\}\,.$$

If $\lim_\omega \frac{\mathrm{dist}(e_n,A_n)}{d_n}=+\infty$ then $\lim_\omega(A_n)=\emptyset$.

Properties of asymptotic cones:

1. $\mathrm{Con}_\omega(X;e,d)$ is a complete metric space;
2. every limit subset $\lim_\omega(A_n)$, if non-empty, is closed;
3. if X is geodesic then every asymptotic cone $\mathrm{Con}_\omega(X;e,d)$ is geodesic;
4. an (L,C)–quasi-isometric embedding between two metric spaces $\mathfrak{q}:X\to Y$ gives rise to a bi-Lipschitz map between asymptotic cones

$$\begin{array}{rcl}\mathfrak{q}_\omega:\mathrm{Con}_\omega(X;e,d) & \to & \mathrm{Con}_\omega(Y;\mathfrak{q}(e),d)\\ \lim_\omega(x_n) & \to & \lim_\omega(\mathfrak{q}(x_n))\,;\end{array}$$

   If moreover $\mathfrak{q}$ is a quasi-isometry then $\mathfrak{q}_\omega$ is onto and one-to-one.

5. If G is a group then every $\mathrm{Con}_\omega(G;e,d)$ is isometric to $\mathrm{Con}_\omega(G;1,d)$, where 1 denotes here the constant sequence equal to 1;
6. $\mathrm{Con}_\omega(G;1,d)$ is a homogeneous space.

It is proved in [dDW84] that any asymptotic cone of a metric space is complete. The same proof gives that $\lim_\omega(A_n)$ is always a closed subset of the asymptotic cone $\mathrm{Con}_\omega(X;e,d)$. Proofs of the previous properties can also be found in [Gro93], [KL97], [KL95]. None of them is difficult; they are good exercises in order to get familiar with the notion. We shall take a closer look only at the last property, namely we shall exhibit a group acting transitively by isometries on $\mathrm{Con}_\omega(G;1,d)$.

6.2 Ultrapowers and internal sets

Definition 6.2 (ultraproduct)**.** For every set X the *ultrapower* X^ω corresponding to an ultrafilter ω consists of equivalence classes of sequences $(x_n)_{n\in\mathbb{N}}$, $x_n\in X$, where two sequences (x_n) and (y_n) are identified if $x_n=y_n$ ω-almost surely. The equivalence class of a sequence $x=(x_n)$ in X^ω is denoted by $x^\omega=(x_n)^\omega$.

Note that any structure on X (group, ring, order, total order) defines a similar structure on X^ω. When $X=\mathbb{K}$ is either $\mathbb{N},\mathbb{Z}$ or $\mathbb{R}$, the ultrapower $\mathbb{K}^\omega$ is sometimes called the *nonstandard extension of* $\mathbb{K}$, and the elements in $\mathbb{K}^\omega\setminus\mathbb{K}$ are called *nonstandard elements.*

Let G be a group. The ultrapower G^ω is also a group. Let $\mathcal{S}_{(1)}(G)$ be the set of sequences defined as in (11). The subgroup $G_1^\omega=\mathcal{S}_{(1)}(G)/\omega$ of G^ω acts transitively by isometries on $\mathrm{Con}_\omega(G;1,d)$ by:

$$(g_n)^\omega\lim_\omega(x_n)=\lim_\omega(g_nx_n)\,.$$

Every subset A of X can be embedded into X^ω by $a \mapsto (a)^\omega$. We denote its image by $\widehat{A}$. We denote the image of each element $a \in A$ by $\widehat{a}$.

Definition 6.3 (internal subsets)**.** A subset A_ω of the ultrapower X^ω is *internal* if there exists a sequence of subsets (A_n) such that $x^\omega = (x_n)^\omega \in A_\omega$ if and only if $x_n \in A_n$ ω-almost surely. We write $A_\omega = (A_n)^\omega$.

Lemma 6.4. *If A is an infinite subset in X then $\widehat{A}$ is not internal.*

Proof. Assume by contradiction that there exists a sequence (A_n) such that $\widehat{A} = (A_n)^\omega$. Let $\{a_1, a_2, ..., a_k, ...\}$ be countably many pairwise distinct elements in A. Define a sequence (x_n) as follows:

- if $\{i \in \mathbb{N} \,;\, a_i \in A_n\}$ has a maximum i_n take $x_n = a_{i_n}$;
- if the maximum above does not exist, consider the minimum j_n of $\{j \in \mathbb{N} \,;\, a_{n+j} \in A_n\}$, and take $x_n = a_{n+j_n}$.

We have that $(x_n)^\omega \in (A_n)^\omega = \widehat{A}$, hence there exists a_k such that $x_n = a_k$ ω-almost surely. Then ω-almost surely the second case cannot occur, and ω-almost surely k is the maximal i such that $a_i \in A_n$. Then $a_{k+1} \notin A_n$ ω-almost surely, hence $a_{k+1} \notin \widehat{A}$, contradiction. □

Definition 6.5 (internal maps)**.** A map $f_\omega : X^\omega \to Y^\omega$ is *internal* if there exists a sequence of maps $f_n : X_n \to Y_n$ such that $f_\omega(x^\omega) = (f_n(x_n))^\omega$ ω-almost surely.

For instance given a metric space (X, dist) one can define a metric dist_ω on X^ω as the internal function $\mathrm{dist}_\omega : X^\omega \times X^\omega \to R^\omega$ defined by the constant sequence of functions (dist).

Note that the range of an internal map is an internal set.

Let (Π) be a property of a structure on the space X that can be expressed using elements, subsets, $\in, \subset, \subseteq, =$ and the logical quantifiers $\exists, \forall, \wedge$ (and), $\vee$ (or), $\neg$ (not) and $\Rightarrow$ (implies).

The non-standard interpretation $(\Pi)^\omega$ of (Π) is the statement obtained by replacing "$x \in X$" with "$x^\omega \in X^\omega$", and "A subset of X" with "A^ω internal subset of X^ω".

Theorem 6.6 (Łoś' Theorem [BS69], [Kei76], Chapter 1, [dDW84], p.361)**.** *A property (Π) is true in X if and only if its non-standard interpretation $(\Pi)^\omega$ is true in X^ω.*

Corollary 6.7.

1. *Every non-empty internal subset in $\mathbb{R}^\omega$ that is bounded from above (below) has a supremum (infimum).*
2. *Every non-empty internal subset in $\mathbb{N}^\omega$ that is bounded from above (below) has a maximal (minimal) element.*

Corollary 6.8 (non-standard induction)**.** *If a non-empty internal subset A^ω in $\mathbb{N}^\omega$ satisfies the properties:*

- $\widehat{1} \in A^\omega$*;*
- *for every $n^\omega \in A^\omega$, $n^\omega + 1 \in A^\omega$;*

then $A^\omega = \mathbb{N}^\omega$.

Exercise 6.9.

1. Deduce Corollary 6.7 from Theorem 6.6.
2. Deduce Corollary 6.8 from Corollary 6.7.

6.3 Asymptotic cones of hyperbolic spaces

Hyperbolic spaces and groups display a remarkable range of properties, as first pointed out by M. Gromov in [Gro87]. It is natural therefore to study their asymptotic cones. The first result in this direction was stated by M. Gromov in [Gro93, §2.A].

Proposition 6.10 (§2.A, [Gro93]). *(i) If the metric space (X, dist) is geodesic and hyperbolic, then every asymptotic cone of it is a real tree.*

(ii) Let (X, dist) be a geodesic metric space. If there exists ω such that for every (x_n) and (d_n), the asymptotic cone $\mathrm{Con}_\omega(X, (x_n), (d_n))$ is a real tree, then (X, dist) is hyperbolic.

We give a short proof of the Proposition. It relies on the following lemma.

Lemma 6.11. *Let (X, dist) be a geodesic metric space. If there exists ω such that for every (x_n) and (d_n), the asymptotic cone $\mathrm{Con}_\omega(X, (x_n), (d_n))$ is a real tree then there exists $M > 0$ such that for every geodesic triangle Δ of vertices x, y, z with $\mathrm{dist}(y, z) \geq 1$ we have that the Hausdorff distance*

$$\mathrm{dist}_H([x, y], [x, z]) \leq M \mathrm{dist}(y, z).$$

Proof. If we suppose the contrary of the conclusion, then there exist sequences of points x_n, y_n, z_n, such that $\mathrm{dist}(y_n, z_n) \geq 1$ and $M_n = \frac{\mathrm{dist}_H([x_n,y_n],[x_n,z_n])}{\mathrm{dist}(y_n,z_n)} \to \infty$. Up to a change of notations we may suppose there exists $a_n \in [x_n, y_n]$ such that $\delta_n = \mathrm{dist}(a_n, [x_n, z_n]) = \mathrm{dist}_H([x_n, y_n], [x_n, z_n]) = M_n \mathrm{dist}(y_n, z_n)$. Since $\delta_n \geq M_n$ it follows $\delta_n \to \infty$.

In the asymptotic cone $\mathrm{Con}_\omega(X, (a_n), (\delta_n))$, the limit sets of $[x_n, y_n]$ and $[x_n, z_n]$ are at Hausdorff distance 1, so they do not coincide.

The triangle inequalities imply that the limits $\lim_\omega \frac{\mathrm{dist}(y_n,a_n)}{\delta_n}$ and $\lim_\omega \frac{\mathrm{dist}(z_n,a_n)}{\delta_n}$ are either both finite or both infinite. It follows that the limit sets of $[x_n, y_n]$ and $[x_n, z_n]$ are either two distinct segments joining the points $[x_n]$ and $[y_n] = [z_n]$, or two distinct asymptotic rays with common origin, or two distinct geodesics asymptotic on both sides. All these cases are impossible in a real tree. □

Proof of Proposition 6.10. The statement **(i)** is obvious. We prove by contradiction that the condition in **(ii)** is sufficient for hyperbolicity. Suppose the geodesic space X is not hyperbolic. Then for every $n \in \mathbb{N}$ there exists a geodesic triangle Δ_n of vertices x_n, y_n, z_n, and there exists $a_n \in [x_n, y_n]$ such that $d_n = \mathrm{dist}(a_n, [x_n, z_n] \cup [y_n, z_n]) \geq n$. Up to a change of notations we may suppose that a_n has been chosen on Δ_n such that the minimum of the distances to the two sides of Δ_n not containing it is maximal. We may likewise suppose that $d_n = \mathrm{dist}(a_n, [y_n, z_n]) = \mathrm{dist}(a_n, b_n)$, where $b_n \in [y_n, z_n]$. Then $\delta_n = \mathrm{dist}(a_n, [x_n, z_n]) = \mathrm{dist}(a_n, c_n) \geq d_n$, where $c_n \in [x_n, z_n]$.

In the asymptotic cone $\mathbf{K} = \mathrm{Con}_\omega(X, (a_n), (d_n))$ we look at the limit set of Δ_n. There are two cases.

A) $\lim_\omega \frac{\delta_n}{d_n} < +\infty$.

By Lemma 6.11, we have $\mathrm{dist}_H([a_n, x_n], [c_n, x_n]) \leq M \cdot \delta_n$. Therefore the limit sets of $[a_n, x_n]$, $[c_n, x_n]$ are either two geodesic segments with a common endpoint or two asymptotic rays. The same is true of the pairs of segments $[a_n, y_n]$, $[b_n, y_n]$ and $[b_n, z_n], [c_n, z_n]$, respectively. It follows that the limit set $[\Delta_n]$ is a geodesic triangle Δ of vertices $x, y, z \in \mathbf{K} \cup \partial_\infty \mathbf{K}$. The point $a = [a_n] \in [x, y]$ is such that $\mathrm{dist}(a, [x, z] \cup [y, z]) \geq 1$, which implies that Δ is not a tripod. This contradicts the fact that $\mathbf{K}$ is a real tree.

B) $\lim_\omega \frac{\delta_n}{d_n} = +\infty$.

This also implies that $\lim_\omega \frac{\text{dist}(a_n,x_n)}{d_n} = +\infty$ and $\lim_\omega \frac{\text{dist}(a_n,z_n)}{d_n} = +\infty$.

By Lemma 6.11, we have $\text{dist}_H([a_n, y_n], [b_n, y_n]) \leq M \cdot d_n$. Thus, the segments $[x_n, y_n]$ and $[y_n, z_n]$ have as limit sets two rays of origin $y = [y_n]$ or two complete geodesics asymptotic on one side. We denote them xy and yz, respectively, with $y \in \mathbf{K} \cup \partial_\infty \mathbf{K}, x, z \in \partial_\infty \mathbf{K}$. The limit set of $[x_n, z_n]$ disappeared from our sight.

The choice of a_n implies that any point of $[b_n, z_n]$ must be at a distance at most d_n from $[x_n, y_n] \cup [x_n, z_n]$. This implies that all points of the ray bz are at distance at most 1 from xy. It follows that xy and yz are either asymptotic rays of origin y or complete geodesics asymptotic on both sides, and they are at Hausdorff distance 1. We again obtain a contradiction of the fact that $\mathbf{K}$ is a real tree.

We conclude that the hypothesis in **(ii)** implies $\exists \delta > 0$ such that X is δ–hyperbolic.

Proposition 6.10 implies that hyperbolicity is invariant up to quasi-isometry. This is due to the fact that a real tree may be defined as a geodesic 0-hyperbolic metric space, or equivalently as a geodesic metric space in which all topological arcs joining two fixed points have the same image. But in fact for the quasi-isometry equivalence it suffices to prove that in a geodesic 0-hyperbolic metric space every bi-Lipschitz arc has the same image as the geodesic joining its endpoints.

Lemma 6.12. *Let X be a geodesic 0-hyperbolic space. Then every bi-Lipschitz arc has the same image as the geodesic joining its endpoints.*

Proof. The 0-hyperbolicity implies that in every polygon one edge is contained in the union of the other edges. Let $\mathfrak{p}$ be a bi-Lipschitz arc with the same endpoints a, b as a geodesic $\mathfrak{g}$. For an arbitrary ϵ let $x_0 = a, x_1, ..., x_n = b$ be points on $\mathfrak{p}$ splitting it into sub-arcs of length at most ϵ. Then the arc $\mathfrak{p}$ and the polygonal line $[x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n]$ are at Hausdorff distance at most ϵ from each other. By 0-hyperbolicity we have that $\mathfrak{g} \subset [x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n] \subset \mathcal{N}_{2\epsilon}(\mathfrak{p})$ for every ϵ. It follows that $\mathfrak{g} \subset \mathfrak{p}$, hence $\mathfrak{g} = \mathfrak{p}$. □

Theorem 6.13 ([DP01]). *Every asymptotic cone of a non-elementary hyperbolic group is isometric to the $2^{\aleph_0}$–universal real tree as defined by Theorem 2.37.*

6.4 Asymptotic cones of groups with polynomial growth

This section follows closely [dDW84], [KL95] and also [Gro81].

Proposition 6.14. *Let (R_n) be a sequence in $\mathbb{R}_+$ diverging to ∞ such that $\mathfrak{G}_G(R_n) \leq CR_n^d$ for some constants $C > 0$ and $d \in \mathbb{N}$. Then there exists $(d_n)^\omega \in \mathbb{R}^\omega$ with $d_n \in [\log R_n, R_n]$ such that for every $i \in \mathbb{N}$, $i \geq 4$,*

$$\text{If } g_1, ..., g_t \text{ are elements in } B(1, d_n/4) \text{ and } B(g_i, d_n/i) \text{ are pairwise disjoint then } t \leq i^{d+1}\ \omega\text{–a.s.} \tag{12}$$

Proof. Assume that (12) is false. Then for any sequence (d_n) with $d_n \in [\log R_n, R_n]$ there exists $i \in \mathbb{N}$, $i \geq 4$, such that for every $n \in \mathbb{N}$ there exist at least i^{d+1} elements in $B(1, d_n/4)$ at distance at least $2d_n/i$ from each other. This allows to define the maps $f_n : [\log R_n, R_n] \to \mathbb{N}$, $f_n(\delta) =$ the minimal $i \in \mathbb{N}$, $i \geq 4$, such that there exist at least i^{d+1} elements in $B(1, \delta/4)$ at distance at least $2\delta/i$ from each other. Let f_ω be the internal map defined by this sequence. According to the hypothesis above, for every (d_n) there exists $i \in \mathbb{N}$ such that $f_n(d_n) \leq i$, that is $f_n(d_n) \in \{1, 2, ..., i\}$. According to the Remark 3.28 it then follows that there exists $j \in \{1, 2, ..., i\}$ such that $f_n(d_n) = j$ ω-almost surely.

Thus $f_\omega : [\log R_n, R_n]^\omega \to \widehat{\mathbb{N}}$. Thus the range of f_ω is $\widehat{A}$ for some $A \subset \mathbb{N}$, and it is internal. Lemma 6.4 implies that A is finite. Take K its maximal element. Note that its minimal element is at least 4.

We prove by non-standard induction (i.e. using Corollary 6.8) that for every $u^\omega \in \mathbb{N}^\omega$ there exists a map $\iota : \{k^\omega \in \mathbb{N}^\omega \; ; \; \widehat{1} \leq k^\omega \leq u^\omega\} \to \{4, ...K\}$ such that either ω-almost surely $\frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega - 1)} \leq \log R^\omega$ or ω-almost surely there exists $t^\omega \in \mathbb{N}^\omega$, $t^\omega \geq \left(\iota(\widehat{1})....\iota(u^\omega)\right)^{d+1}$ and points $x_{\widehat{1}}, ..., x_{t^\omega} \in B_{X^\omega}(1, R^\omega/4)$ at pairwise distance at least $\frac{2R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}$ from each other.

For $u^\omega = \widehat{1}$ we know from the above that there exists $\iota(\widehat{1}) = j \in \{4, ...K\}$ such that there exist at least j^{d+1} elements in every $B(1, R_n/4)$ at distance at least $2R_n/j$ from each other.

Assume that the statement is true for $u^\omega \in \mathbb{N}^\omega$. If ω-almost surely $\frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)} \leq \log R^\omega$ then one can easily deduce the same for $u^\omega + 1$.

Assume that there exists $t^\omega \in \mathbb{N}^\omega$, $t^\omega \geq \left(\iota(\widehat{1})....\iota(u^\omega)\right)^{d+1}$ and points $x_{\widehat{1}}, ..., x_{t^\omega} \in B_{X^\omega}(1, R^\omega/4)$ at pairwise distance at least $\frac{2R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}$ from each other. In other words the balls $B\left(x_{k^\omega}, \frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}\right)$ with $\widehat{1} \leq k^\omega \leq t^\omega$ are pairwise disjoint. Each nonstandard ball $B\left(x_{k^\omega}, \frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}\right)$ is isometric to $B\left(1, \frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}\right)$. Since $\frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)} \geq \log R^\omega$ it follows that we can find $j \in \{4, ...K\}$ such that there exist at least j^{d+1} elements in every $B\left(1, \frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}\right)$ at distance at least $\frac{2R^\omega}{\iota(\widehat{1})....\iota(u^\omega)j}$ from each other. Thus we define $\iota(u^\omega + 1) = j$.

Consider the set $\mathfrak{S}$ of $u^\omega \in \mathbb{N}^\omega$ for which there exists a map $\iota : \{k^\omega \in \mathbb{N}^\omega \; ; \; \widehat{1} \leq k^\omega \leq u^\omega\} \to \{4, ...K\}$ such that ω-almost surely there exists $t^\omega \in \mathbb{N}^\omega$, $t^\omega \geq \left(\iota(\widehat{1})....\iota(u^\omega)\right)^{d+1}$ and points $x_{\widehat{1}}, ..., x_{t^\omega} \in B_{X^\omega}(1, R^\omega/4)$ at pairwise distance at least $\frac{2R^\omega}{\iota(\widehat{1})....\iota(u^\omega)}$ from each other.

It is bounded from above by b^ω such that $\frac{R^\omega}{4^{b^\omega}} \leq \log R^\omega$, i.e. by $\log R^\omega - \log\log R^\omega$. Then by Corollary 6.7 the set $\mathfrak{S}$ has a maximal element, that is there exists $u^\omega \in \mathfrak{S}$ such that $u^\omega + 1 \notin \mathfrak{S}$. Hence $\frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega+1)} \leq \log R^\omega$, while $\frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega)} > \log R^\omega$. Since $u^\omega \in \mathfrak{S}$, it follows that there exist at least $\left(\iota(\widehat{1})....\iota(u^\omega)\right)^{d+1}$ points in $B_{X^\omega}(1, R^\omega/4)$. The inequality $\log R^\omega \geq \frac{R^\omega}{\iota(\widehat{1})....\iota(u^\omega+1)} \geq \frac{R^\omega}{K\iota(\widehat{1})....\iota(u^\omega)}$ implies that $\iota(\widehat{1})....\iota(u^\omega) \geq \frac{R^\omega}{K \log R^\omega}$. Therefore $B_{X^\omega}(1, R^\omega/4)$ contains at least $\left(\frac{R^\omega}{K \log R^\omega}\right)^{d+1}$ elements.

On the other hand $|B_{X^\omega}(1, R^\omega)| \leq C(R^\omega)^d$. It follows that $R^\omega \leq C(K \log R^\omega)^{d+1}$. This is impossible. $\square$

Proposition 6.15. *Let G be a finitely generated group such that for some sequence (R_n) in $\mathbb{R}_+$ diverging to ∞ we have that $\mathfrak{G}_G(R_n) \leq CR_n^d$ for some constants $C > 0$ and $d \in \mathbb{N}$. Then for the sequence (d_n) provided by Proposition 6.14 the asymptotic cone $\mathrm{Con}_\omega(G; 1, (d_n))$ is proper and of Hausdorff dimension at most $d + 1$.*

Proof. Since $\mathrm{Con}_\omega(G; 1, (R_n))$ is geodesic and complete it suffices by the Hopf-Rinow Theorem 1.5 to prove that it is locally compact. Since it is homogeneous it suffices to prove that the closed ball of radius $1/4$ and centre 1 is compact. By Proposition 6.14 in this ball $\frac{2}{i}$–separated subsets have at most i^{d+1} elements, for every $i \geq 4$.

Lemma 6.16. *Let K be a closed subset in a complete metric space (X, dist) with the property that for every $i \geq 4$ any $\frac{2}{i}$–separated subset of K has at most $N(i)$ elements. Then K is compact.*

Proof. Since we are in a metric space it suffices to prove that K is sequentially compact. Let (x_n) be a sequence in K. Let $y_1, .., y_N$ be a maximal $\frac{1}{2}$–separated subset in K. Then it is also a $\frac{1}{2}$–net. A subsequence $(x_{1,n})$ of (x_n) is entirely contained in a ball $B(y_i, 1/2)$. Inductively, given a subsequence $(x_{k,n})$ entirely contained in a ball of radius $\frac{1}{2^k}$ we find a subsequence $(x_{k+1,n})$ of $(x_{k,n})$ entirely contained in a ball of radius $\frac{1}{2^{k+1}}$. The diagonal sequence $(x_{k,k})$ is easily seen to be Cauchy: for $k, m \geq k_0$, $x_{k,k}$ and $x_{m,m}$ are contained in the same ball of radius $\frac{1}{2^{k_0}}$. Since (X, dist) is complete it follows that $x_{k,k}$ converges to a point x, and since K is closed $x \in K$. □

We recall the definition of the Hausdorff measure and dimension. For details see [Rog98] or [Fal90].

Let (X, dist) be a metric space and let $\alpha > 0$. The α–Hausdorff measure $H_\alpha(A)$ of a subset A of X is

$$H_\alpha(A) = \lim_{r \to 0} \inf \left\{ \sum_{i \in I} r_i^\alpha \; ; \; I \text{ countable, } X = \bigcup_{i \in I} B(x_i, r_i) \text{ for some } x_i \in X \right\} .$$

It is a countably additive measure, but it may take value $+\infty$. It may even be constant $+\infty$ or constant 0.

The Hausdorff dimension of a subset A in X is $\inf\{\alpha \; ; \; H_\alpha(A) \equiv 0\}$.

Theorem 6.17. *The covering dimension of a metric space is smaller or equal to its Hausdorff dimension.*

Lemma 6.18. *Let (X, dist) be a metric space with the property that for every $i \geq 4$ any $\frac{2}{i}$–separated subset of X has at most i^{d+1} elements. Then X has Hausdorff dimension at most $d+1$.*

Proof. Take $\alpha > d+1$, and take a maximal $\frac{2}{i}$–separated subset of X. Then $H_\alpha(X) \leq \lim_{i \to \infty} (2/i)^\alpha \cdot i^{d+1} = 0$. □

Lemma 6.18 and Proposition 6.14 also finish the proof of Proposition 6.15. □

The following result is an immediate consequence of the Montgomery-Zippin theory [MZ74] which provided an answer to Hilbert's fifth problem (to characterize Lie groups among topological groups).

Theorem 6.19. *([MZ74], 6.3, [Are46], p. 606) Let X be a metric space which is complete, connected, locally connected, proper and of finite covering dimension. If its group of isometries $Isom(X)$ acts transitively on X then $Isom(X)$ is a Lie group with finitely many connected components (the induced topology being the compact-open topology).*

A *Lie group* is a group with a structure of differentiable manifold such that both the product and the inversion are smooth maps. The standard Lie theory implies that for every Lie group L there exists a homomorphism $L \to GL(n, \mathbb{C})$ with kernel in the centre of L. This allows to extend the Tits alternative as follows:

Theorem 6.20 (Tits' alternative [Tit72]). *Let G be a finitely generated subgroup of a Lie group with finitely many connected components. Then either G has a non-Abelian free subgroup or G is virtually solvable.*

7 Proof of the Polynomial Growth Theorem of M. Gromov.

Theorem 7.1. *([Gro81]) A finitely generated group with polynomial growth is virtually nilpotent.*

Proof. We prove by induction on d that if G is a finitely generated group with $\mathfrak{G}_G(x) \preceq x^d$ then G is virtually nilpotent.

Remark 7.2. In fact it becomes clear from the proof that it is enough to require that $\mathfrak{G}_G(R_n) \preceq R_n^d$ for some sequence $R_n \to \infty$.

For $d = 0$ the group G must be finite. Assume that the statement is true for d and let G be a finitely generated group with $\mathfrak{G}_G(x) \preceq x^{d+1}$. By Proposition 6.15, an asymptotic cone $\mathbf{K} = \mathrm{Con}_\omega(G; 1, (d_n))$ is complete, geodesic, proper, of covering dimension at most $d+1$. Then $Isom(\mathbf{K})$ is a Lie group with finitely many connected components, and there exists an embedding $G \to \widehat{G} \subseteq G_1^\omega \subseteq Isom(\mathbf{K})$, which we denote by φ.

According to Theorem 6.20, either $\varphi(G)$ contains a free non-Abelian subgroup or it is virtually solvable. In the former case it follows that G contains a free non-Abelian subgroup, which contradicts the hypothesis that G has polynomial growth. It follows that $\varphi(G)$ is virtually solvable. Up to exchanging G with a finite index subgroup we may assume that $\varphi(G)$ is solvable.

If $\varphi(G)$ is infinite solvable then there exists an onto homomorphism $\varphi(G) \to \mathbb{Z}$, hence an onto homomorphism $G \to \mathbb{Z}$. Proposition 5.20 implies that its kernel N is finitely generated and $\mathfrak{G}_N(x) \preceq x^d$. The inductive hypothesis implies that N is a virtually nilpotent group. Lemma 5.19 implies that $G \simeq N \rtimes \mathbb{Z}$, and Proposition 5.16 implies that G is virtually nilpotent.

Assume that $\varphi(G)$ is finite. For instance when G is Abelian we even have $\varphi(G) = \{\mathrm{id}\}$. Up to replacing G with the kernel of φ, a subgroup of finite index in G, we may assume that $\varphi(G) = \{\mathrm{id}\}$. Let $S = \{s_1, ..., s_n\}$ be a finite set of generators of G and let dist be the metric on the Cayley graph $\mathrm{Cayley}(G, S)$. Given an element $p \in G$, we define $\delta(g; p, R) = \max\{\mathrm{dist}(gx, x)\ ;\ x \in B(p, R)\}$, and $\Delta(G; p, R) = \max_{s\in S} \delta(s; p, R)$.

Lemma 7.3. *If the function $R \mapsto \Delta(G; 1, R)$ is bounded then G is virtually Abelian.*

Proof. The hypothesis implies that there exists a constant $C > 0$ such that for every $s \in S$ and $x \in G$, $\mathrm{dist}(sx, x) \leq C \Leftrightarrow x^{-1}sx \in B(1, C)$. Let $|B(1, C)| = N$. Then the centralizer of s in G, $Z_G(s) = \{x \in G\ ;\ xs = sx\}$ has index at most N in G. Indeed given F a collection of $N + 1$ elements in G, $x^{-1}sx$, $x \in F$, is contained in $B(1, C)$, hence there exist $x, y \in F$, $x \neq y$, such that $x^{-1}sx = y^{-1}sy$, hence such that $y^{-1}x \in Z_G(s)$.

The intersection $\bigcap_{s\in S} Z_G(s)$ has finite index in G and it is Abelian. □

We may henceforth assume that the function $R \mapsto \Delta(G; 1, R)$ is unbounded, and that G is not virtually Abelian.

Lemma 7.4. *For every $\lambda \in (0, 1]$ there exists $x_n \in G$ such that*

$$\lim_\omega \frac{\Delta(G; x_n, d_n)}{d_n} = \lambda\,. \tag{13}$$

Proof. Since $\lim_\omega \frac{\Delta(G;1,d_n)}{d_n} = 0$ it follows that ω-almost surely $\Delta(G; 1, d_n) < \lambda d_n$. On the other hand, since $R \mapsto \Delta(G; 1, R)$ is unbounded there exists y_n such that $\Delta(G; y_n, d_n) > \lambda d_n$. Since $\mathrm{Cayley}(G, S)$ is connected and the map $y \mapsto \Delta(G; y, d_n)$ is continuous it follows there exists $z_n \in \mathrm{Cayley}(G, S)$ such that $\Delta(G; z_n, d_n) = \lambda d_n$. If z_n is in the interior of an edge, take y_n a vertex at distance at most 1/2 of it. Then $|\Delta(G; y_n, d_n) - \lambda d_n| \leq 1$, whence (13). □

For every $\lambda \in (0,1]$ we consider the element $x_n \in G$ provided by Lemma 7.4 and define the homomorphism $\varphi_\lambda : G \to G_1^\omega$ by $\varphi_\lambda(g) = (x_n^{-1} g x_n)^\omega$. If for some $\lambda \in (0,1]$, $\varphi_\lambda(G)$ is infinite then we finish the argument as previously.

Assume that for all $\lambda \in (0,1]$, $\varphi_\lambda(G)$ is finite. For every $s \in S$, $\delta(s;1,1) \leq \lambda$, and there exists $s \in S$ such that $\delta(s;1,1) = \lambda$. This implies that $\varphi_\lambda(S)$ is contained in smaller and smaller neighbourhoods of the identity element in L as $\lambda \to 0$. A Lie group has the property that for every $m \in \mathbb{N}$ there exists a neighbourhood of the identity element which does not contain elements of order $\leq m$. It follows that $\max_{s\in S} \mathrm{order}\varphi_\lambda(s) \to \infty$ as $\lambda \to 0$. In particular $|\varphi_\lambda(G)| \to \infty$ as $\lambda \to 0$. Now we use the following theorem.

Theorem 7.5 (Jordan's Theorem [Rag72], Theorem 8.29)**.** *Let L be a Lie group with finitely many connected components. Then there exists $N = N(L)$ such that every finite subgroup F in L contains an Abelian subgroup of index at most N.*

Thus if in G we consider G_1 the intersection of all the subgroups of G of index at most N (which is also a finite index subgroup) we have that $\varphi_\lambda(G_1)$ is Abelian for all $\lambda \in (0,1]$. Also $|\varphi_\lambda(G_1)| \to \infty$ as $\lambda \to 0$. According to Exercise 5.1, each homomorphism φ_λ defines an onto homomorphism $\overline{\varphi}_\lambda : (G_1)_{ab} \to \varphi_\lambda(G_1)$. It follows that $(G_1)_{ab}$ is infinite, in particular $(G_1)_{ab} \simeq A_f \times \mathbb{Z}^k$, where A_f is finite Abelian and $k \geq 1$. This allows to define an onto homomorphism $G_1 \to \mathbb{Z}$ and to finish the argument as previously. □

The complete result relating polynomial growth function and topological properties of the asymptotic cones is the following.

Theorem 7.6. *Let (G, dist) be a finitely generated group with a word metric.*

a) *$\mathrm{Con}_\omega(G;1,(d_n))$ is a proper metric space for all ω and $(d_n) \iff (G,\mathrm{dist})$ has polynomial growth $\iff G$ is virtually nilpotent.*

Suppose in the sequel that G is nilpotent and each $C^iG/C^{i+1}G \simeq A_i^f \times \mathbb{Z}^{m_i}$.

b) *All the asymptotic cones of G are isometric, and isometric to a graded Lie group G_∞ endowed with a Carnot–Caratheodory metric (see [Pan83] for a definition of the group G_∞ and of its metric);*

c) *The covering dimension of the limit space is equal to $\sum_i m_i$.*

d) *The Hausdorff dimension of the limit space is equal to the degree of polynomial growth and to $\sum_i (i+1)m_i$.*

Morality: Outside the class of virtually nilpotent groups one should not expect the asymptotic cones to be locally compact.

The second equivalence in statement a) of the previous theorem is the Polynomial Growth Theorem of M. Gromov, and it implies that virtual nilpotency is a quasi-isometry invariant in the class of finitely generated groups. The direct part of statement a) is proved in [Dru02].

The equality of d) is the result of H. Bass [Bas72].

Statement d) emphasizes that the degree of polynomial growth, if it is not infinite, it is a quasi-isometry invariant.

The statement b) as well as the first equalities in c) and d) are proved by P. Pansu [Pan83].

8 Dictionary

- **(Abstractly) commensurable groups.** Two discrete groups G_1 and G_2 are called *abstractly commensurable* if they have finite index subgroups that are isomorphic.

- **Action by isometries.** An action $\alpha : G \times X \to X$ of a group G on a metric space (X, dist_X) is *an action by isometries* if for every $g \in G$ the map

  $$\alpha_g : X \to X \,,\; \alpha_g(x) = \alpha(g, x)\,,$$

  is an isometry.

- **Boundary at infinity.** Given X either a simply connected Riemannian manifold of non-positive curvature (or more generally a $CAT(0)$–space) or an infinite graph, its boundary at infinity $\partial_\infty X$ is the quotient $\mathcal{R}/\sim$ of the set $\mathcal{R}$ of geodesic rays in X with respect to the equivalence relation $r_1 \sim r_2 \Leftrightarrow \mathrm{dist}_H(r_1, r_2) < +\infty$.

- **Bounded metric space, bounded subset in a metric space.** A metric space is *bounded* if it has finite diameter (see below for a definition).

  A subset A of a metric space (X, dist) is *bounded* if endowed with the induced metric it is bounded; equivalently if $A \subseteq B(x_0, r)$ for some $x_0 \in X$ and $r > 0$.

- **Closure of a set (in a topological space).**Given a topological space X and a subset $A \subseteq X$, the *closure of A in X*, denoted by $\overline{A}$, is the smallest (with respect to inclusion) closed subset of X containing A, that is $\overline{A}$ is closed in X, and if $A \subseteq V$ where V is closed in X then $\overline{A} \subseteq V$. Equivalently $\overline{A} = \bigcap_{F \text{closed}, A \subseteq F} F$.

  Another equivalent definition of the closure is

  $$\overline{A} = \{x \in X : U \cap A \neq \emptyset \quad \text{for any open } \; U \subseteq X \quad \text{with } \; x \in U\}.$$

  A point in $\overline{A}$ is sometimes called an *adherent point of A*.

- **Commensurable groups in an ambient larger group.** When both G_1 and G_2 are subgroups in a group G, we say that G_1 and G_2 are *commensurable (in G)* if there exists $g \in G$ such that $G_1^g \cap G_2$ has finite index both in G_1^g and in G_2.

- **Compact set.** A subset A of a topological space X such that every open cover for A (i.e. family $\{U_i : i \in I\}$ of open subsets of X such that $A \subseteq \bigcup_{i \in I} U_i$) has a finite subcover (i.e. a subfamily $\{U_1, ..., U_m\} \subset \{U_i : i \in I\}$ such that $A \subseteq \bigcup_{i=1}^{m} U_i$).

  In a metric space a set is compact if and only if it is *sequentially compact*, i.e. every sequence in that set has a subsequence converging to a point in that set.

  A compact subset of a metric space is closed, bounded and complete.

- **Complete metric space.** A metric space in which every Cauchy sequence converges (to a point in the space). A sequence (x_n) in a metric space (X, dist) is *Cauchy* if given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mathrm{dist}(x_m, x_n) < \varepsilon$ whenever $m \geqslant n \geqslant N$.

- **Diameter of a metric space.** Given a metric space (X, dist) its *diameter* is

$$\mathrm{diam}(X) = \sup\{\mathrm{dist}(x,y) \; ; \; x,y \in X\}\,.$$

Possibly $\mathrm{diam}(X) = +\infty$. A metric space with finite diameter is said to be *bounded*.

- **Direct product.** A group G is a direct product of two *normal* subgroups N and N' if and only if one of following equivalent statements holds:

  - $G = NN' = \{nn' \; ; \; n \in N \, , \, n' \in N'\}$ and $N \cap N' = \{1\}$;
  - $G = N'N$ and $N \cap N' = \{1\}$;
  - for every element g of G there exists a unique $n \in N$ and $n' \in N'$ such that $g = nn'$;
  - for every element g of G there exists a unique $n \in N$ and $n' \in N'$ such that $g = n'n$.

  Conversely, given any two groups N and N' (not necessarily subgroups of the same group), one can define a new group $G = N \times N'$ which is a direct product of a copy of N and a copy of N' in the above sense, defined as follows. As a set, $N \times N'$ is defined as the cartesian product $N \times N'$. The binary operation $*$ on G is defined by

$$(n_1, n_1') * (n_2, n_2') = (n_1 n_2, n_1' n_2')\,,\ \forall n_1, n_2 \in N \text{ and } n_1', n_2' \in N'\,.$$

  The group G is a direct product of $N \times \{1\}$ and $\{1\} \times N'$ in the sense above. The group $G = N \times N'$ is called the *direct product of N and N'*.

  If a group G is the direct product of two normal subgroups N and N' then G is isomorphic to $N \times N'$.

- **Distance between subsets.** If A and B are two subsets in a metric space X, then the distance $\mathrm{dist}(A,B)$ between A and B is the infimum over all $\mathrm{dist}(a,b)$ with $a \in A$ and $b \in B$.

  If A, B are both compact then there exists $a \in A$ and $b \in B$ such that $\mathrm{dist}(a,b) = \mathrm{dist}(A,B)$. The same holds if A is compact, B is closed and the ambient metric space is proper.

- **Elementary group.** It is a group that contains a cyclic subgroup (possibly trivial) of finite index.

- **Free action.** Action of a group G on a space X such that the stabilizer of each point x, $Stab(x) = \{g \in G \; ; \; gx = x\}$ is the trivial sub-group $\{1\}$.

- **Fuchsian group.** It is a discrete subgroup of $PSL(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$.

- **(Global) cut-point.** A point p in a topological space X such that $X \setminus \{p\}$ has several connected components.

- **Hausdorff distance.** If A and B are two subsets in a metric space X, then the Hausdorff distance $\mathrm{dist}_H(A,B)$ between A and B is the minimum of all $\delta > 0$ such that A is contained in the δ-tubular neighborhood of B and B is contained in the δ-tubular neighborhood of A. If no such finite δ exists, one puts $\mathrm{dist}_H(A,B) = +\infty$.

- **Isometric embedding. Isometry.** A map $f : X \to Y$ between two metric spaces (X, dist_X) and (Y, dist_Y) is an *isometric embedding* if

$$\mathrm{dist}_Y(f(x_1), f(x_2)) = \mathrm{dist}_X(x_1, x_2)\,.$$

  If moreover f is onto then f is called an *isometry.*

- **Locally compact space.** A topological space with the property that for every point x in it there exists an open subset U containing x, and contained in a compact subset (equivalently, such that $\overline{U}$ is compact).

- **Metric space.** It is a non-empty set X endowed with a function $\mathrm{dist} : X \times X \to \mathbb{R}$ with the following properties:

(M1) $\mathrm{dist}(x, y) \geqslant 0$ for all $x, y \in X$; $\mathrm{dist}(x, y) = 0$ if and only if $x = y$;

(M2) (Symmetry) for all $x, y \in X$, $\mathrm{dist}(y, x) = \mathrm{dist}(x, y)$;

(M3) (Triangle inequality) for all $x, y, z \in X$, $\mathrm{dist}(x, z) \leqslant \mathrm{dist}(x, y) + \mathrm{dist}(y, z)$.

  The function dist is called *metric* or *distance.*

- **Proper (discontinuous) action of a discrete group**. The action by homeomorphisms of a discrete group G on a topological space such that for every compact subset K of X the set $\{g \in G\ ;\ gK \cap K \neq \emptyset\}$ is finite.

  If X is locally compact and G acts properly (discontinuously) on X then $G\backslash X$ is also locally compact, and Hausdorff.

- **Proper metric space**. A metric space with the property that all its closed balls are compact. Equivalently, all its bounded closed sets are compact.

  Note that a proper metric space is locally compact and complete. A partial converse is the Hopf-Rinow Theorem [Gro07, §1.10] every complete, locally compact length metric space is proper.

- **Reduced words.** Given an alphabet $S = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$, a word in the alphabet S is called *reduced* if it does not contain subwords of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$.

- **Semidirect product.** A group G is a semidirect product of two subgroups N and H, which is sometimes denoted by $G = N \rtimes H$ if and only if N is a *normal subgroup* of G, H is a *subgroup* of G, and one of following equivalent statements holds:

  - $G = NH = \{nh\ ;\ n \in N\,,\ h \in H\}$ and $N \cap H = \{1\}$;
  - $G = HN$ and $N \cap H = \{1\}$;
  - for every element g of G there exists a unique $n \in N$ and $h \in H$ such that $g = nh$;
  - for every element g of G there exists a unique $n \in N$ and $h \in H$ such that $g = hn$;
  - there exists a homomorphism $G \to H$ which restricted to H is the identity on H and whose kernel is N.

Let $\mathrm{Aut}(N)$ denote the group of all automorphisms of N. The map $\varphi : H \to \mathrm{Aut}(N)$ defined by $\varphi(h)(n) = hnh^{-1}$, is a group homomorphism.

Conversely, given any two groups N and H (not necessarily subgroups of the same group) and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$, one can define a new group $G = N \rtimes_\varphi H$ which is a semidirect product of a copy of N and a copy of H in the above sense, defined as follows. As a set, $N \rtimes_\varphi H$ is defined as the cartesian product $N \times H$. The binary operation $*$ on G is defined by

$$(n_1, h_1) * (n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2)\,,\ \forall n_1, n_2 \in N \text{ and } h_1, h_2 \in H\,.$$

The group G is a semidirect product of $N \times \{1\}$ and $\{1\} \times H$ in the sense above. The group $G = N \rtimes_\varphi H$ is called the *semidirect product of N and H with respect to φ*.

If a group G is the semidirect product of a normal subgroup N with a subgroup H then G is isomorphic to $N \rtimes_\varphi H$, where

$$\varphi(h)(n) = hnh^{-1}\,.$$

If φ is the trivial homomorphism, sending every element of H to the identity automorphism of N, then $N \rtimes_\phi H$ is the direct product $N \times H$.

- **Subgroup generated by a set.** If S is a subset of a group G, then $< S >$, the *subgroup generated by S*, is the smallest subgroup of G containing every element of S, meaning the intersection over all subgroups containing the elements of S; equivalently, $< S >$ is the subgroup of all elements of G that can be expressed as the finite product of elements in S and their inverses.

  If S is the empty set, then $< S >= e$, since we consider the empty product to be the identity.

  When there is only a single element x in S, $< S >$ is usually written as $< x >$; it is the cyclic subgroup of the powers of x.

- **Tubular neighborhood**. For a set A in a metric space X and for $\delta > 0$ we define the *δ-tubular neighborhood $\mathcal{N}_\delta(A)$ of A* as the set

$$\{x \mid \mathrm{dist}(x, A) < \delta\}\,.$$

- **Virtually *.** A group is said to have property * *virtually* if a finite index subgroup has the property *.

- **Zorn's Lemma.** Every non-empty partially ordered set in which every totally ordered subset (chain) has an upper bound contains at least one maximal element.

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