Quasi-isometry invariants and asymptotic cones

Cornelia DRUŢU*

UFR de Mathématiques et UMR CNRS 8524,
Université de Lille I, F–59655 Villeneuve d'Ascq, France

Abstract

We present some results about quasi-isometry invariants, particularly emphasizing the relationships between their behavior and properties of asymptotic cones. Our main objects of study are hyperbolic metric spaces, hyperbolic and solvable groups.

1 Introduction

The study of quasi-isometry invariants is a part of a larger and more complex program formulated by M. Gromov [Gr_1]: to study the behavior of infinite finitely generated groups with respect to quasi-isometry. We shall discuss here several quasi-isometry invariants, without attempting to be exhaustive. The emphasis will be on the relation between quasi-isometry invariants and properties of asymptotic cones.

A *quasi-isometry* between two metric spaces (X, d) and (Y, d') is a map $q : X \to Y$ such that

$$\frac{1}{L}d(x_1, x_2) - c \leq d'(q(x_1), q(x_2)) \leq Ld(x_1, x_2) + c,\ \forall x_1, x_2 \in X,$$

for some fixed positive constants c and L, and such that Y is at finite Hausdorff distance from the image of q. We recall that the Hausdorff distance between two subsets A and B of a metric space X, which we denote by $d_H(A, B)$, is the infimum of the values of δ such that $A \subset V_\delta(B)$ and $B \subset V_\delta(A)$.

Notation: $V_\delta(A)$ denotes the set $\{x \in X \mid d(x, A) \leq \delta\}$.

The Hausdorff distance between two subsets of a metric space can be infinite. It is a real distance only for certain classes of subsets (for instance for the class of compact subsets).

Two metric spaces, X and Y, are called *quasi-isometric* if there is a quasi-isometry between them. A classical example of quasi-isometric spaces is that of the universal cover $\widetilde{M}$ of a cocompact Riemann manifold M and of the fundamental group $\pi_1(M)$ endowed with a word metric with respect to a finite generating set.

*drutu@agat.univ-lille1.fr

A property that is invariant with respect to quasi-isometry is called *a geometric property*. Sometimes we shall consider only metric spaces in a certain class (for instance only groups of a certain kind with a word metric). A property of a space in that class invariant with respect to quasi-isometry between two spaces in the class is called *a geometric property in the class*. For instance hyperbolicity as defined in Section 3.A is a geometric property in the class of geodesic metric spaces [Gr_3]. This is also a consequence of Proposition 3.A.1, (b), below. Also, the property of being virtually abelian or virtually nilpotent (see Section 5) or of being amenable [GH_2] are geometric properties in the class of finitely generated groups. We recall that a group is said to have a certain property *virtually* if it has a finite index subgroup having that property.

We say that a certain class of finitely generated groups is *closed under quasi-isometry* if every finitely generated group quasi-isometric to a group in the class is virtually a group in the class.

The paper is organized as follows. In Section 2 we recall different ways of defining a limit space of a metric space, as well as the definitions and some properties of the quasi-isometry invariants we shall deal with.

In Section 3 we first present results on limit spaces of hyperbolic metric spaces and of hyperbolic groups. Then we show how much of the behavior of certain quasi-isometry invariants for hyperbolic spaces can be deduced from the structure of the limit spaces. In the end, we present examples due to S. Thomas and B. Velickovic [ThV] of groups which have two non-homeomorphic asymptotic cones. These examples rely on results in the first part of the section.

Section 4 is devoted to a discussion of the known theorems relating the simple connectedness of the asymptotic cones to a polynomial filling order.

The fact that among all finitely generated groups only virtually nilpotent groups have proper asymptotic cones, as well as many other results on quasi-isometry invariants for virtually nilpotent groups, are discussed in Section 5.

In the final section we discuss results about quasi-isometries and quasi-isometry invariants of solvable groups.

2 Preliminaries

2.A Limit spaces

Given a metric space (X, d), we present some ways to define a limit for the sequences of metric spaces $\left(X, \frac{1}{d_n}d\right)$, where (d_n) is a sequence of positive numbers, $d_n \to \infty$. The aim of each notion of limit is that of giving an image of the metric space "seen from infinitely far away", thus suppressing all bounded features and leaving the parts of the geometry which determine the large scale behavior.

The first attempt to do this is by means of a distance. M. Gromov defines the *modified Hausdorff pseudo-distance* (also called the *Gromov-Hausdorff pseudo-distance*) [Gr_1, §6] on the class of proper metric spaces, as follows:

$$d_{Hm}((X,d_X),(Y,d_Y)) = \inf_{(x,y)\in X\times Y} \inf\{\varepsilon > 0 \mid \exists\, d \text{ metric on } X \vee Y \text{ such that } d(x,y) < \varepsilon,$$

$$d|_{X\times X} = d_X, d\,|_{Y\times Y}\; d_Y \text{ and } B(x,1/\varepsilon) \subset V_\varepsilon(Y), B(y,1/\varepsilon) \subset V_\varepsilon(X)\}.$$

Notation: $B(x,R)$ denotes the ball of center x and radius R.

For the homogeneous spaces the modified Hausdorff pseudo-distance coincides with:

$$d_{\tilde{H}}((X,d_X,x_0),(Y,d_Y,y_0)) = \inf\{\varepsilon > 0 \mid \exists\, d \text{ metric on } X \vee Y \text{ such that}$$
$$d(x_0,y_0) < \varepsilon,\ d\,|_{X\times X}= d_X, d\,|_{Y\times Y}= d_Y,\ B(x_0,1/\varepsilon) \subset V_\varepsilon(Y), B(y_0,1/\varepsilon) \subset V_\varepsilon(X)\},$$

which is called the *modified Hausdorff pseudo-distance (or Gromov-Hausdorff pseudo-distance) with distinguished points.*

Example: Let $X = \mathbb{R}$ with the canonical metric and let Y be the planar circle of radius r, $\mathcal{C}(O,r)$, with the length metric. Both are homogeneous spaces, so the modified Hausdorff pseudo-distance between them coincides with the modified Hausdorff pseudo-distance with distinguished points. We show that $d_{\tilde{H}}((\mathbb{R}\,,\,0),(\mathcal{C}(O,r)\,,\,N)) = \frac{4}{\sqrt{\pi^2r^2+16}+\pi r}$, where N is the north pole of the circle.

We glue the interval $\left[-\frac{\pi}{2}r\,,\,\frac{\pi}{2}r\right]$ in $\mathbb{R}$ isometrically to half of the circle, so that 0 is identified to N, and we endow the graph thus obtained with its length metric. The minimal ε such that in this metric $\left[-\frac{1}{\varepsilon}\,,\,\frac{1}{\varepsilon}\right] \subset V_\varepsilon(\mathcal{C}(O,r))$ and $B(N,1/\varepsilon) \subset V_\varepsilon(\mathbb{R})$ is $\varepsilon_0 = \frac{4}{\sqrt{\pi^2r^2+16}+\pi r}$. This value is the positive solution of the equation

$$(E) \qquad\qquad \frac{\pi}{2}r + \varepsilon = \frac{1}{\varepsilon}\,.$$

Let d' be another metric on $\mathbb{R}\vee\mathcal{C}(O,r)$ which coincides with their length metrics on both $\mathbb{R}$ and $\mathcal{C}(O,r)$. Let ε' be the smallest $\varepsilon > 0$ such that $d'(0,N) < \varepsilon$ and $\left[-\frac{1}{\varepsilon}\,,\,\frac{1}{\varepsilon}\right] \subset V_\varepsilon(\mathcal{C}(O,r)),\ B(N,1/\varepsilon) \subset V_\varepsilon(\mathbb{R})$ in the metric d'. Let x', y' be the nearest points in $\mathcal{C}(O,r)$ to $-\frac{1}{\varepsilon'}$ and $\frac{1}{\varepsilon'}$, respectively. Since $d'(x',y') \leq \pi r$, it follows that $\frac{2}{\varepsilon'} \leq \pi r + 2\varepsilon'$. The previous inequality implies that $\varepsilon' \geq \varepsilon_0$.

The modified Hausdorff pseudo-distance with distinguished points satisfies the triangle inequality if restricted to a class of metric spaces "$d_{\tilde{H}}$-near" one to the other (for instance to a class $\mathcal{X}$ of metric spaces such that for every $X, Y \in \mathcal{X},\ d_{\tilde{H}}(X,Y) \leq \frac{1}{2}$). We also note that if Y is a dense subset in X and $x_0 = y_0$ then $d_{\tilde{H}}((X,d_X,x_0),(Y,d_Y,y_0)) = 0$. Thus, in order for $d_{\tilde{H}}$ to become a true distance, we also have to restrict it to a class of proper metric spaces.

A definition of the limit space of a sequence of type $(X,\varepsilon_n d)$, $\varepsilon_n \to 0$, with the modified Hausdorff pseudo-distance is possible in some cases. Only we have to ask that the limit should be proper, otherwise it is not well defined: it may be replaced by any dense subset of it. But the condition that the limit space should be proper is a very restrictive condition, as shown in Section 5.

We recall two other definitions of "limit spaces" of a metric space: the asymptotic cone and the asymptotic subcone. The former has been used for the first time by M. Gromov in [Gr$_2$]. Van den Dries and A. J. Wilkie [VDW] used nonstandard analysis tools to give a precise definition of the notion. Their definition was resumed by M. Gromov in [Gr$_4$], where he discussed many examples and many properties related to asymptotic cones. The notion of asymptotic subcone has been defined by M. Gromov in [Gr$_3$].

A filter on a set X is a set $\mathfrak{F}$ of subsets of X with the following properties:

(F1) If $A \subset B \subset X$ and $A \in \mathfrak{F}$ then $B \in \mathfrak{F}$;

(F2) If $A_i \in \mathfrak{F},\ i = 1, 2, \dots n$, then $\cap_{i=1}^n A_i \in \mathfrak{F}$;

(F3) $\emptyset \notin \mathfrak{F}$.

Example: Suppose X is infinite. The set of complements of the finite subsets of X is a filter. If $X = \mathbb{N}$ this filter is called *the Fréchet filter*.

A base of a filter on a set X is a set $\mathfrak{B}$ of subsets of X which satisfies the properties:

(B1) If $B_i \in \mathfrak{B},\ i = 1, 2$, then $B_1 \cap B_2$ contains a subset of $\mathfrak{B}$;

(B2) $\emptyset \notin \mathfrak{B}$ and $\mathfrak{B}$ is not empty.

If $\mathfrak{B}$ is a base of a filter, then the set of subsets of X containing an element of $\mathfrak{B}$ is a filter.

An ultrafilter on a set X is a filter $\mathfrak{U}$ which is a maximal element in the ordered set of all filters on X with respect to the inclusion. An ultrafilter $\mathfrak{U}$ has the important property that if $A \cup B \in \mathfrak{U}$, then either $A \in \mathfrak{U}$ or $B \in \mathfrak{U}$ [Bou, §I.6.4].

The axiom of choice implies that every filter is contained in at least one ultrafilter [Bou, §I.6.4]. In particular, the Fréchet filter is contained in an ultrafilter on $\mathbb{N}$. An ultrafilter on $\mathbb{N}$ containing the Fréchet filter is called *a non-principal ultrafilter*. Let $\mathfrak{U}$ be such an ultrafilter. The map $\omega = \mathbf{1}_{\mathfrak{U}}$ is a finitely additive probability measure $\omega : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ with value 0 on all finite sets. We shall use the same name, of non-principal ultrafilter, for $\mathfrak{U}$ and for ω.

Remark 2.A.1. *(1) Let $\mathcal{S} = \{s_1, s_2, \dots, s_n, \dots\}$ be an infinite subset of $\mathbb{N}$. There exists a nonprincipal ultrafilter ω such that $\omega(\mathcal{S}) = 1$.*

(2) Let $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \cdots \supset \mathcal{S}_m \supset \dots$ be an infinite sequence of infinite subsets of $\mathbb{N}$. There exists a nonprincipal ultrafilter ω such that $\omega(\mathcal{S}_m) = 1$, $\forall m \in \mathbb{N}$.

Proof. (1). Let $\mathfrak{F}$ be the Fréchet filter. Let $\mathcal{S}^k = \{s_k, s_{k+1}, \dots, s_n, \dots\}$. Then $\mathfrak{B} = \mathfrak{F} \cup \{\mathcal{S}, \mathcal{S}^2, \mathcal{S}^3, \dots, \mathcal{S}^k, \dots\}$ is a filter base. There exists therefore an ultrafilter $\mathfrak{U}$ containing it and we take $\omega \mathbf{1}_{\mathfrak{U}}$.

(2). We argue as previously, with the filter base

$$\mathfrak{B}' = \mathfrak{F} \cup \bigcup_{m=1}^{\infty} \{\mathcal{S}_m, \mathcal{S}_m^2, \mathcal{S}_m^3, \dots, \mathcal{S}_m^k, \dots\}.$$

$\diamondsuit$

If a set $\mathcal{S} \subset \mathbb{N}$ has the property that $\omega(\mathcal{S}) = 1$, then $\omega(\complement\mathcal{S}) = 0$. It follows that in all arguments done ω-almost surely, only what happens for the integers in $\mathcal{S}$ counts. According to Remark 2.A.1, any infinite subset of $\mathbb{N}$ is the set of integers that "count" in the arguments done ω-almost surely for some nonprincipal ultrafilter ω.

Let ω be a fixed nonprincipal ultrafilter. In a topological space, *the ω-limit* of a sequence, (a_n), is defined as an element a such that for every neighborhood $\mathcal{N}(a)$ of a, $\omega(\{n \in \mathbb{N} \mid a_n \in \mathcal{N}(a)\}) = 1$. We denote a by $\lim_\omega a_n$. The ω-limit is unique in any Hausdorff separated space. Any sequence in a compact space has an ω-limit, for any non-principal ultrafilter ω [Bou, I.9.1].

Let (X, d) be a metric space. Let (x_n) be a sequence of points in X, which we call *sequence of observation centers*. Let (d_n) be a sequence of positive numbers diverging to infinity, which we call *sequence of scalars*, and let ω be a non-principal ultrafilter. We define $\mathcal{C}$ to be the set of sequences (y_n) of points in X with the property that $\frac{d(x_n,y_n)}{d_n}$ is bounded. Consider the following equivalence relation defined on $\mathcal{C}$:

$$(y_n) \sim (z_n) \Leftrightarrow \lim_{\omega} \frac{d(y_n, z_n)}{d_n} = 0\,.$$

The quotient space $X_\omega((x_n), (d_n)) = \mathcal{C}/\sim$ is called *the asymptotic cone of X with respect to the observation centers (x_n), the scalars (d_n) and the non-principal ultrafilter ω*. It is a complete metric space with the metric

$$D([y_n], [z_n]) = \lim_{\omega} \frac{d(y_n, z_n)}{d_n}\,.$$

Moreover, $X_\omega((x_n), (d_n))$ is geodesic if X is geodesic.

We shall call *quasi-homogeneous space* a metric space X such that the action of $Isom\,(X)$ on X has a bounded fundamental domain. If moreover there exists a discrete subgroup of isometries of X acting properly discontinuously with compact quotient on X, the space is called *cocompactly homogeneous*.

Remark 2.A.2. *Let X be a quasi-homogeneous space, and let $x_0 \in X$ be fixed. Let ω and (d_n) be fixed. Then all $X_\omega((x_n), (d_n))$ are isometric to the asymptotic cone $X_\omega((x_0), (d_n))$.*

By means of ω, we may define another quotient space $X^* = X^{\mathbb{N}}/\approx$, where the equivalence relation $\approx$ is defined by:

$$(x_n) \approx (y_n) \Leftrightarrow x_n = y_n\ \omega-\text{almost surely.}$$

The space X^* is called *the ultrapower of X over the ultrafilter ω.*

We denote by $[[x_n]]$ the equivalence class of the sequence (x_n) with respect to the equivalence relation $\approx$.

Let π^* be the canonical map from $X^{\mathbb{N}}$ onto X^*. Denote $\mathcal{C}^* = \pi^*(\mathcal{C})$. We have the sequence of surjective maps:

$$\mathcal{C} \xrightarrow{\pi^*} \mathcal{C}^* \xrightarrow{\pi} X_\omega\,, \text{ where } \quad \pi([[x_n]]) = [x_n]\,.$$

A subset A of X^* is called *internal set* if there exists a sequence of sets $A_n \subset X$ such that $A = \{[[x_n]] \mid x_n \in A_n\ \omega\text{-a.s.}\}$. We write $A = [[A_n]]$.

A subset A of X_ω is called *limit set* if there exists a sequence of sets $A_n \subset X$ such that $A = \{[x_n] \mid x_n \in A_n\ \omega\text{-a.s.}\}$. We write $A = [A_n]$.

Another notion of limit space is that of *asymptotic subcone*. For a metric space (X, d), an asymptotic subcone is a geodesic metric space (X_∞, d_∞) with the property that for every finite set of points in it, $\{x_\infty^1, x_\infty^2, \dots, x_\infty^k\}$, there exist a sequence $\varepsilon_n \to 0$ and a sequence of finite k-sets in X, $\{x_n^1, x_n^2, \dots, x_n^k\}$, such that

$$\lim_{n\to\infty} \varepsilon_n d(x_n^i, x_n^j) = d_\infty(x_\infty^i, x_\infty^j),\ \forall i, j \in \{1, 2, \dots, k\}\,.$$

Concerning the relationship between the three previous definitions of limit of a metric space, the following can be said. Let (X,d) and (Y,δ) be proper metric spaces. If the sequence of metric spaces with distinguished points $(X,(1/d_n)d,x_n)$ converges to (Y,δ,y_0) in the modified Hausdorff metric, then for all ultrafilters ω, there exists an isometry between $X_\omega((x_n),(d_n))$ and (Y,y_0) such that the image of $[x_n]$ is y_0 [KaL$_1$].

The converse is not true in general. The hypothesis that for (x_n) and (d_n) fixed and for all ultrafilters ω, $X_\omega((x_n),(d_n))$ are isometric, so isometric to a model space (Y,δ,y_0), does not imply that $(X,(1/d_n)d,x_n)$ converges to (Y,δ,y_0) in the modified Hausdorff metric (see Section 3.C). The implication becomes true if one adds the hypothesis that one $X_\omega((x_n),(d_n))$ is a proper metric space (see Section 5).

The notion of asymptotic subcone is much weaker than that of asymptotic cone. Every asymptotic cone is an asymptotic subcone, but the converse is not true. This is due to the fact that in the definition of an asymptotic subcone only the behavior of finite configurations of points is taken into account. For instance, given a hyperbolic non-elementary group Γ, all its asymptotic cones are isometric and are complete real trees with all points branching points with a continuum order of branching (see Section 3.A for details). On the other hand, it is not difficult to see that any tree, whether finite, simplicial or real, is an asymptotic subcone of Γ.

In this paper we shall concentrate on some properties of the asymptotic cones and subcones for hyperbolic metric spaces (groups) in the sense of Gromov and for solvable groups. Also, we shall discuss some relationships between properties of the asymptotic cones and behavior of quasi-isometry invariants, and their applications. We mention here that the asymptotic cones are used to prove the quasi-isometry invariance of the geometric decomposition of fundamental groups of Haken 3-manifolds and to describe groups quasi-isometric to such a fundamental group ([KaL$_1$], [KaL$_2$]) and to prove the quasi-isometric rigidity of uniform lattices [KlL] as well as of non-uniform lattices [Dr$_5$] in higher rank semisimple groups.

2.B Quasi-isometry invariants

We may define a notion of "generic loop", of "disk filling a loop" and of "area of a disk filling a loop" in three different settings: Riemannian manifolds, finitely presented groups and more generally geodesic metric spaces (see [Gr$_3$, §2.3], [Gr$_4$, §5.F], [Bow$_2$, §2.3], [BH, Chapter I, §8A.4]). In the sequel we recall the meaning of these notions in geodesic metric spaces, as given by M. Gromov in [Gr$_4$]. Let X be such a space and let $\delta > 0$ be a fixed constant. By a "generic loop" we mean a Lipschitz map $c : \mathbb{S}^1 \to X$. We denote by Ω the set of all generic loops. *A filling partition of c* is a pair consisting of a triangulation of the planar unit disk $\mathbb{D}^2$ and of an injective map, $\sigma : \wp \to X$, from the set of vertices of the triangulation to X, which extends $c|_{\wp\cap\mathbb{S}^1}$. The image D of the map σ is called *filling disk of c*. We can join the images of vertices of each triangle by geodesics, except for the vertices which are ends of arcs of $\mathbb{S}^1$. Here, instead of the geodesic, we leave the arc of $c(\mathbb{S}^1)$ contained between the images. We call the triangles thus obtained *bricks of the partition*. Sometimes we also call the partition a *partition of c into n bricks*, where n is the number of bricks. Let $\mathcal{G}$ be the graph composed of the bricks of the partition. We endow the filling disk D with a metric d_D defined as $d_D(x,y)$ =the length of the shortest path in $\mathcal{G}$ joining x and y.

The length of a brick is the sum of the distances between vertices, with the same exception as before, where we replace the distance by the length of the arc of $c(\mathbb{S}^1)$ contained between the two images of vertices. *The mesh of the partition*, $Mesh(\sigma)$, is the maximum of the lengths of the bricks. We denote by $Mesh(c,n)$ the minimal mesh of a partition of c into n bricks.

The partition σ is called a *δ-filling partition of c* if $Mesh(\sigma) \leq \delta$. The filling disk D is then called a *δ-filling disk of c*. For a δ-filling disk of c, *the area of the disk* is the number of bricks of the partition. We denote by $P(c,\delta)$ the minimal number of bricks of a δ-partition of c, which is the same thing as the minimal area of a δ-filling disk of c.

The *δ-area function* is defined as $Ar_\delta : \Omega \to \mathbb{R}_+^* \cup \{+\infty\},\ Ar_\delta(c) = P(c,\delta)$.

We may likewise define the *δ-radius function* as $r_\delta : \Omega \to \mathbb{R}_+^* \cup \{+\infty\},\ r_\delta(c) = \inf\{\max_{x\in D} d_D(x, c(\mathbb{S}^1)) \mid D \text{ is a } \delta - \text{filling disk of the loop } c\}$.

Both functions depend on the parameter δ, and may take infinite values. In order to obtain finite valued functions we add the hypothesis that there exists μ big enough such that for $\delta \geq \mu$, every loop has a δ-filling disk. We call a metric space satisfying this hypothesis a *μ-simply connected* metric space. All metric spaces we are interested in (Cayley graphs of finitely presented groups, Riemannian manifolds, Euclidean buildings etc.) are μ-simply connected, for appropriate values of μ. In the sequel we shall only deal with μ-simply connected metric spaces. We shall omit to recall this hypothesis each time. We shall see that the dependence on δ of the area and the radius function is not essential.

For Riemannian manifolds and finitely presented groups, we can also define an area function and a radius function in a classical manner, using smooth filling disks and disks whose bricks are translates of relators, respectively. In all cases the area function satisfies the following property, as shown by B. Bowditch in [Bow_2, §2.3 and chapter 5]:

(quadrangle inequality) Let $c \in \Omega$ and let $c(\mathbb{S}^1) = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ be a splitting into four consecutive arcs. Then, with the notations $d_1 = d(\alpha_1, \alpha_3)$ and $d_2 = d(\alpha_2, \alpha_4)$ we have

$$Ar(c) \geq k d_1 d_2 \,,$$

where k is a universal constant.

The area function allows to define the *filling function* as

$$A : \mathbb{R}_+^* \to \mathbb{R}_+^*,\ A(\ell) = \sup\{Ar(c) \mid c \text{ loop of length } \leq \ell\}.$$

The previous function is also called *Dehn function* for finitely presented groups. We note that Dehn functions are defined on and take values in $\mathbb{N}$. For our convenience, in the case of geodesic metric spaces we also use the notation $P(\ell,\delta)$ for $\sup\{P(c,\delta) \mid c \text{ of length at most } \ell\}$. In fact it is the same thing as $A(\ell)$, which we should in this case denote by $A_\delta(\ell)$. But for $A_\delta(\ell)$ the constant δ is usually supposed fixed, therefore we shall prefer the notation $P(\ell,\delta)$ each time we have to use different δ in our arguments.

Likewise, using the radius function we may define the *filling radius function* as

$$r : \mathbb{R}_+^* \to \mathbb{R}_+^*,\ r(\ell) = \sup\{r(c) \mid c \text{ loop of length } \leq \ell\} \,.$$

We recall an equivalence relation between real valued functions. Let $f_i : D_i \to C$ ($i = 1,2$) be two real valued functions of real variable. *The order of the function f_1 is at most the order of the function f_2*, relation which we denote by $f_1 \prec f_2$, if there exists a constant $c > 0$ such that $f_1(t) \le c\tilde{f_2}(ct) + ct$, $\forall t \in D_1$, where $\tilde{f_2}(t) = \sup\{f_2(x) \mid x \in D_2 \text{ and } x \le t\}$. The functions f_1 and f_2 are said to have *the same order*, which we denote by $f_1 \stackrel{\circ}{=} f_2$, if $f_1 \prec f_2$ and $f_2 \prec f_1$. The relation $\stackrel{\circ}{=}$ is an equivalence relation. The equivalence class of a numerical function with respect to this relation is called *the order of the function.* If a function f has (at most) the same order as the function x, x^2, x^3, x^d or $\exp x$ it is said that *the order of the function f is (at most) linear, quadratic, cubic, polynomial, or exponential,* respectively.

In a geodesic metric μ-simply connected space X, if $\delta_i > 0$, $i = 1,2$, are big enough, then $A_{\delta_1}(\ell) \stackrel{\circ}{=} A_{\delta_2}(\ell)$ and $r_{\delta_1}(\ell) \stackrel{\circ}{=} r_{\delta_2}(\ell)$.

If two metric spaces X_i, $i = 1,2$, are μ_i-simply connected and quasi-isometric, then their filling functions have the same order. The same is true for their filling radius functions (see [Al], [Ge$_1$], [BMS], [Dr$_2$]). In particular, if a finitely presented group acts by isometries properly discontinuously and with compact quotient on a simply connected Riemannian manifold then the Dehn function of the group and the filling function of the manifold have the same order [BT].

We should note here that the property of being finitely presented is a geometric property among finitely generated groups [GH$_2$]. Therefore, as soon as we may talk about Dehn function for a group we may do it for all finitely generated groups quasi-isometric to it.

The order of the filling function of a metric space X is also called *the filling order of X*. Thus, the filling order of a space and the order of the filling radius are quasi-isometry invariants. This is one of the reasons why they are interesting to study. Another reason is given by the following result, a proof of which can be found for instance in [Ge$_1$].

Proposition 2.B.1. *In a finitely presented group G the following statements are equivalent.*

(S_1) *G has a solvable word problem.*

(S_2) *the Dehn function of G is recursive.*

(S_3) *the filling radius function of G is recursive.*

If in a metric space X the filling function $A(\ell)$ satisfies $A(\ell) \prec \ell$ or ℓ^2 or e^ℓ, it is said that the space X *satisfies a linear, quadratic or exponential isoperimetric inequality.*

The following theorem of M.Sapir, J.-C.Birget, E.Rips, which is very close to a complete description of all Dehn functions $\succ n^4$, shows that the class of Dehn functions of finitely presented groups is very large. It equally shows that the class of the filling radius functions is very large.

Theorem 2.B.2 ([SBR]). *(1) Every Dehn function of a finitely presented group has the same order as the time function of a (not necessarily deterministic) Turing machine.*

(2) Let $T(n)$ be the time function of a Turing machine such that $T(n)^4$ is superadditive. Then there exists a finitely presented group with Dehn function of the same order as $T(n)^4$ and filling radius function of the same order as $T(n)^3$.

Corollary 2.B.3 ([SBR]). *(1) Let α be a real number bigger than 4 such that the k-th decimal fraction of α is computable in time $T_k \prec 2^{2^k}$. Then there exists a finitely presented group with Dehn function of the same order as $f(n) = n^\alpha$ and filling radius function of the same order as $r(n) = n^{\frac{3}{4}\alpha}$.*

(2) If $f(n) = n^\alpha$ is of the same order as the Dehn function of a finitely presented group, then the k-th decimal fraction of α can be computed in time $T_k \prec 2^{2^{2^k}}$.

The set IP of α such that $f(n) = n^\alpha$ is of the same order as the Dehn function of a finitely presented group, is sometimes called *the isoperimetric spectrum*. Corollary 2.B.3 implies that $IP \cap [4, \infty)$ contains many of the known numbers: e^2, $\pi + 1$, $e\pi$, the algebraic numbers, all values at integer points of functions which are solutions of differential equations with integral polynomial coefficients etc. The inclusion $\mathbb{N} \subset IP$ has been proved previously ([BMS], [BP],[Gr_4]). M. Gromov showed that $IP \cap (1, 2) = \emptyset$. It is not difficult to see that $IP \cap (0, 1) = \emptyset$ (see Section 3.B for details on the two previous statements). In [Br_2] it has been shown that there exist infinitely many numbers in $(\mathbb{Q} \setminus \mathbb{Z}) \cap (2, \infty) \cap IP$. Also, in [BB] it is proved that for all pairs of positive integers $p > q$, the number $2 \log_2(2p/q)$ is in IP. This in particular implies that the closure of IP is $\{1\} \cup [2, \infty)$.

The Dehn function/filling function can be generalized to higher dimensions and give n-Dehn functions, which give information about the way to fill topological spheres $\mathbb{S}^n$ with topological balls $\mathbb{B}^{n+1}$ ([Gr_4, Chapter 5], [ECHPT, Chapter 10], [P_6]). The following result holds.

Proposition 2.B.4 ([$\mathbf{P}_6$]). *The second Dehn function of a group of type $\mathbf{FP}_3$ is bounded by a recursive function.*

The condition $\mathbf{FP}_3$ is a higher dimensional variant of the condition of finite presentation in a group. The latter is equivalent to the existence of a finite collection $\mathcal{L}$ of loops in the Cayley graph such that every other loop has a filling disk whose bricks are images under G of elements of $\mathcal{L}$. If one glues the boundary of a disk onto every loop in $\mathcal{L}$ one obtains a simply connected cell (and even simplicial) complex with 1-skeleton the Cayley graph. The condition $\mathbf{FP}_3$ means that there exists a finite collection $\mathfrak{S}$ of topological 2-spheres in this complex such that every other 2-sphere can be filled with their translations under G. We call translations of elements of $\mathfrak{S}$ *basic spheres*.

The idea in the proof of Proposition 2.B.4 is to produce an algorithm which, given $n \in \mathbb{N}$, finds in finite time an upper bound on the number of basic spheres needed to fill any 2-dimensional sphere composed of at most n cells. Such an algorithm exists because for a group Γ of presentation $< S \mid R >$ and of type $\mathbf{FP}_3$ one can recognize the 2-dimensional spheres in the simplicial complex constructed as before: they are the 2-dimensional disks whose boundary is labelled by a word in S composed, up to cyclic permutations, of pairs ss^{-1}, $s \in S$. For each 2-dimensional sphere, one has only to check all the possible fillings.

The previous algorithm does not work for 1-dimensional spheres, that is, loops. The reason is that there is no way to recognize the paths in the Cayley graph of Γ (labelled by words in S) which are loops. Thus, in applying the previous algorithm, one has to consider all words of length at most n, and the risk is to try to find the "filling" for a path which is not a loop, in which case the algorithm never ends.

The algorithm only gives a recursive bound of the second Dehn function, because the filling found by it might be not the smallest possible.

Let (X, d) be a cocompactly homogeneous metric space. Let Γ be a group of isometries acting properly discontinuously cocompactly on X and let x be a fixed point in X. *The growth function of* X is the function $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(r) = \text{card}\,\{\gamma \in \Gamma \mid \gamma x \in B(x, r)\}$. The order of this function does not depend on the fixed point x, it depends only on the quasi-isometry class of X. This implies in particular that the growth function of X has the same order as the growth function of Γ endowed with a word metric.

If X is a Riemannian manifold, we may also define a growth function with basepoint x as $\overline{f}(r) = vol(B(x, r))$. A growth function defined in this way has the same order as a growth function defined as previously, for cocompactly homogeneous Riemannian manifolds. This has been first noticed by A. S. Švarč [Šv].

3 Hyperbolicity and asymptotic (sub)cones

3.A Results

The property of hyperbolicity for metric spaces is a generalization due to M. Gromov of the property of having negative sectional curvature for Riemannian manifolds.

A metric space (X, d) is called δ-hyperbolic, $\delta > 0$, if for some(every) point $a \in X$ we have

$$(x.y)_a \geq \min\{(x.z)_a, (y.z)_a\} - \delta, \quad \forall x, y, z \in X,$$

where $(x.y)_a = \frac{1}{2}(d(a, x) + d(a, y) - d(x, y))$ is the so-called *Gromov inner product between x and y with respect to a.*

In a geodesic metric space (X, d) the previous definition is equivalent, for a different δ, to the property that for every geodesic triangle, each side of the triangle is contained in the δ-neighborhood of the two other sides.

A geodesic 0-hyperbolic metric space is called *real tree.* It can also be defined as a geodesic metric space such that any two points of it are the endpoints of a unique topological arc.

The property of hyperbolicity is invariant up to quasi-isometry. In particular the fundamental group of a compact Riemannian manifold of negative curvature is hyperbolic. Hyperbolic groups display a remarkable range of properties, as first pointed out by M. Gromov in [Gr_3]. It is natural therefore to study their asymptotic cones. The first result in this direction was stated by M. Gromov in [Gr_4, §2.A].

Proposition 3.A.1 (§2.A, [Gr_4]). *(i) If the metric space (X, d) is hyperbolic, then every asymptotic cone of it is 0-hyperbolic. If (X, d) is moreover geodesic, then every asymptotic cone of it is a real tree.*

(ii) Let (X, d) be a geodesic metric space. If there exists ω such that for every (x_n) and (d_n), the asymptotic cone $X_\omega((x_n), (d_n))$ is a real tree, then (X, d) is hyperbolic.

(iii) Let (X,d) be a geodesic metric space. If for every ω and (x_n), the asymptotic cone $X_\omega((x_n),(n))$ is a real tree, then (X,d) is hyperbolic.

The statements **(ii)** and **(iii)** of Proposition 3.A.1 imply that hyperbolicity is invariant up to quasi-isometry. We give a short proof of the Proposition. It relies on the following lemma.

Lemma 3.A.2. *Let (X,d) be a geodesic metric space. If one of the following two conditions holds*

(a) there exists ω such that for every (x_n) and (d_n), the asymptotic cone $X_\omega((x_n),(d_n))$ is a real tree;

(b) for every ω and (x_n), the asymptotic cone $X_\omega((x_n),(n))$ is a real tree;

then there exists $M>0$ such that for every geodesic triangle Δ of vertices x, y, z with $d(y,z)\geq 1$ we have

$$d_H([x,y],[x,z])\leq Md(y,z).$$

Proof. If we suppose the contrary of the conclusion, then there exist sequences of points x_n, y_n, z_n, such that $d(y_n,z_n)\geq 1$ and $M_n=\frac{d_H([x_n,y_n],[x_n,z_n])}{d(y_n,z_n)}\to\infty$. Up to a change of notations we may suppose there exists $a_n\in[x_n,y_n]$ such that $\delta_n=d(a_n,[x_n,z_n])=d_H([x_n,y_n],[x_n,z_n])=M_nd(y_n,z_n)$. Since $\delta_n\geq M_n$ it follows that $\delta_n\to\infty$.

Suppose condition **(a)** holds. In the asymptotic cone $X_\omega((a_n),(\delta_n))$, the limit sets of $[x_n,y_n]$ and $[x_n,z_n]$ are at Hausdorff distance 1, so they do not coincide.

The triangle inequalities imply that the limits $\lim_\omega\frac{d(y_n,a_n)}{\delta_n}$ and $\lim_\omega\frac{d(z_n,a_n)}{\delta_n}$ are either both finite or both infinite. It follows that the limit sets of $[x_n,y_n]$ and $[x_n,z_n]$ are either two distinct segments joining the points $[x_n]$ and $[y_n]=[z_n]$, or two distinct asymptotic rays with common origin, or two distinct geodesics asymptotic on both sides. All these cases are impossible in a real tree.

Suppose condition **(b)** holds. Let $\mathcal{S}=\{[\delta_n]\mid n\in\mathbb{N}\}$, where $[\delta_n]$ is the integer part of δ_n. By Remark 2.A.1, there exists ω such that $\omega(\mathcal{S})=1$. We consider a sequence (a'_m) satisfying the condition $a'_m=a_n$ if $m=[\delta_n]$. Likewise, we consider sequences (x'_m), (y'_m), (z'_m), satisfying similar conditions with respect to the sequences (x_n), (y_n), (z_n), respectively. If there exist $n\neq n'$ such that $[\delta_n]=[\delta_{n'}]$ then we choose only one set of a_n,x_n,y_n,z_n, to put into the sequences (a'_m), (x'_m), (y'_m), (z'_m) at the index $m=[\delta_n]$.

In $X_\omega((a'_m),(m))$ the limits sets of $[x'_m,y'_m]$ and of $[x'_m,z'_m]$ are in one of the three situations discussed in the previous case, all situations being impossible in a real tree. $\diamondsuit$

Proof of Proposition 3.A.1. First we prove by contradiction that the conditions in **(ii)** and **(iii)** are sufficient for hyperbolicity. Suppose the geodesic space X is not hyperbolic. Then for every $n\in\mathbb{N}$ there exists a geodesic triangle Δ_n of vertices x_n, y_n, z_n, and there exists $a_n\in[x_n,y_n]$ such that $d_n=d(a_n,[x_n,z_n]\cup[y_n,z_n])\geq n$. Up to a change of notations we may suppose that a_n has been chosen on Δ_n such that the minimum of the distances to the two sides of Δ_n not containing it is maximal. We may likewise suppose that $d_n=$

$d(a_n, [y_n, z_n]) = d(a_n, b_n)$, where $b_n \in [y_n, z_n]$. Then $\delta_n = d(a_n, [x_n, z_n]) = d(a_n, c_n) \geq d_n$, where $c_n \in [x_n, z_n]$.

Suppose the hypothesis in **(ii)** holds. In the asymptotic cone $\mathbf{K} = X_\omega((a_n), (d_n))$ we look at the limit set of Δ_n. There are two cases.

A) $\lim_\omega \frac{\delta_n}{d_n} < +\infty$.

By Lemma 3.A.2, we have $d_H([a_n, x_n], [c_n, x_n]) \leq M \cdot \delta_n$. Therefore the limit sets of $[a_n, x_n]$, $[c_n, x_n]$ are either two geodesic segments with a common endpoint or two asymptotic rays. The same is true of the pairs of segments $[a_n, y_n]$, $[b_n, y_n]$ and $[b_n, z_n], [c_n, z_n]$, respectively. It follows that the limit set $[\Delta_n]$ is a geodesic triangle Δ of vertices $x, y, z \in \mathbf{K} \cup \partial_\infty \mathbf{K}$. The point $a = [a_n] \in [x, y]$ is such that $d(a, [x, z] \cup [y, z]) \geq 1$, which implies that Δ is not a tripod. This contradicts the fact that $\mathbf{K}$ is a real tree.

B) $\lim_\omega \frac{\delta_n}{d_n} = +\infty$.

This also implies that $\lim_\omega \frac{d(a_n, x_n)}{d_n} = +\infty$ and that $\lim_\omega \frac{d(a_n, z_n)}{d_n} = +\infty$.

By Lemma 3.A.2, we have $d_H([a_n, y_n], [b_n, y_n]) \leq M \cdot d_n$. Thus, the segments $[x_n, y_n]$ and $[y_n, z_n]$ have as limit sets two rays of origin $y = [y_n]$ or two complete geodesics asymptotic on one side. We denote them by xy and yz, respectively, with $y \in \mathbf{K} \cup \partial_\infty \mathbf{K}, x, z \in \partial_\infty \mathbf{K}$. The limit set of $[x_n, z_n]$ disappeared from our sight.

The choice of a_n implies that any point of $[b_n, z_n]$ must be at a distance at most d_n from $[x_n, y_n] \cup [x_n, z_n]$. This implies that all points of the ray bz are at distance at most 1 from xy. It follows that xy and yz are either asymptotic rays of origin y or complete geodesics asymptotic on both sides, and they are at Hausdorff distance 1. We again obtain a contradiction of the fact that $\mathbf{K}$ is a real tree.

We conclude that the hypothesis in **(ii)** implies $\exists \delta > 0$ such that X is δ-hyperbolic.

Suppose the hypothesis in **(iii)** holds. Let $\mathcal{S} = \{[d_n] \mid n \in \mathbb{N}\}$. Let ω be the nonprincipal ultrafilter such that $\omega(\mathcal{S}) = 1$ (Remark 2.A.1). We consider a sequence of geodesic triangles in X, (Δ'_m), satisfying $\Delta'_m = \Delta_n$ if $m = [d_n]$. If there exist $n \neq n'$ such that $[d_n] = [d_{n'}] = m$ then we choose only one of the geodesic triangles Δ_n with $[d_n] = m$ to put in the sequence (Δ'_m) at the index m. The sequences of vertices of Δ'_m, (x'_m), (y'_m), (z'_m), satisfy similar conditions with respect to the sequences (x_n), (y_n), (z_n), respectively. Also, we may choose a sequence (a'_m) of points on $[x'_m, y'_m]$ such that $a'_m = a_n$ if $m = [d_n]$.

In the asymptotic cone $\mathbf{K}' = X_\omega((a'_m), (m))$, we may consider the limit set of the triangles (Δ'_m), argue as previously, and obtain a contradiction to the fact that $\mathbf{K}'$ is a real tree. It follows that the hypothesis in **(iii)** also implies $\exists \delta > 0$ such that X is δ-hyperbolic.

(i) If X is geodesic the statement is obvious. If X is not geodesic we use the following proposition and the geodesic case. $\diamondsuit$

Proposition 3.A.3 ([Gr$_2$], §6.4.D). *Every hyperbolic space can be isometrically embedded in a geodesic hyperbolic space.*

A variant of Proposition 3.A.1 has been formulated and proved in [GH$_1$].

Proposition 3.A.4 (Proposition 11, Chapter 2, §1, [GH$_1$]). *A metric space is hyperbolic if and only if any asymptotic subcone of it is a real tree.*

A result of the same kind can be found in [Pau]: a sequence of hyperbolic spaces endowed with isometric actions of a fixed group Γ gives in the limit a real tree endowed with an isometric action of Γ.

In the case of a quasi-homogeneous hyperbolic space, X, more can be said on its asymptotic cones. We first recall two definitions from the real tree theory. Let T be a real tree and let p be a point in it. *The space of directions in* p, Σ_p, is defined as $\Re_p/\sim$, where

$$\Re_p = \{r : [0,a) \to T \mid a > 0,\ r \text{ isometry},\ r(0) = p\}$$

and

$$r_1 \sim r_2 \Longleftrightarrow \exists\, \varepsilon > 0 \text{ such that } r_1|_{[0,\varepsilon)} \equiv r_2|_{[0,\varepsilon)}.$$

A *branching point* of T is a point p such that card Σ_p is at least 3. The number card Σ_p is called the *branching number* of the branching point p. The *order of branching* of T is the maximal branching number in T. A real tree is called α-*universal* if every real tree with order of branching at most α can be isometrically embedded into it. See [MNO] for a study of universal trees. In particular, in the previously cited paper it is shown that α-universal trees exist for every α and that for α fixed they are unique up to isometry.

For every homeomorphism $c : [a,b] \to T$ the image $c([a,b])$ coincides with the geodesic segment $[c(a), c(b)]$. It follows that we may also define Σ_p as $\Im_p/\sim$, where

$$\Im_p = \{c : [0,a) \to T \mid a > 0,\ c \text{ homeomorphism},\ c(0) = p\}$$

and

$$c_1 \sim c_2 \Longleftrightarrow \exists\, \varepsilon_1 > 0,\ \varepsilon_2 > 0 \text{ such that } c_1([0,\varepsilon_1)) = c_2([0,\varepsilon_2)).$$

We also recall that a group is called *elementary* if it contains a cyclic subgroup (possibly trivial) of finite index.

Proposition 3.A.5. *Let X be a geodesic hyperbolic space for which there exists a discrete non-elementary group of isometries, acting properly discontinuously, with compact quotient. Then all asymptotic cones $X_\omega((x_n),(d_n))$ are complete real trees where all points are branching points with a continuum order of branching.*

Proof. According to Remark 2.A.2, it suffices to study the asymptotic cones $X_\omega((x_0),(d_n))$, where x_0 is a fixed point. Every asymptotic cone $X_\omega((x_0),(d_n))$ is a real tree.

Let Γ be a discrete non-elementary group of isometries of X, acting properly discontinuously, with compact quotient on X. The existence of Γ implies that X_ω is homogeneous. It suffices therefore to prove that $[x_0]$ is a branching point with a continuum order of branching.

By [GH$_1$], p.60, the group Γ is of finite type and hyperbolic and its Cayley graph $\mathcal{C}(\Gamma)$ is quasi-isometric to X. It follows that there exists a bi-Lipschitz bijection Φ :

$\mathcal{C}(\Gamma)_\omega((e),(d_n)) \to X_\omega((x_0),(d_n)), \Phi([e]) = [x_0]$. The map Φ determines a bijection between $\Sigma_{[e]}$ and $\Sigma_{[x_0]}$. It suffices then to prove that card $\Sigma_{[e]} = 2^{\aleph_0}$ in $\mathcal{C}(\Gamma)_\omega$.

a) We show that the geodesic rays joining e to distinct points of $\partial_\infty \mathcal{C}(\Gamma)$ give distinct directions in $[e]$ in $\mathcal{C}(\Gamma)_\omega$.

Let $r_i : [0,\infty) \to \mathcal{C}(\Gamma)$ be two isometries, $r_i(0) = e, i \in \{1,2\}, r_1(\infty) = \alpha, r_2(\infty) = \beta$, where $\alpha \neq \beta$. For every t and s in $[0,\infty)$, we consider $a_t = [r_1(d_n t)]$ and $b_s = [r_2(d_n s)]$, $a_t,\ b_s \in \mathcal{C}(\Gamma)_\omega$. We have

$$d(a_t,b_s) \lim_\omega \frac{d(r_1(d_n t), r_2(d_n s))}{d_n} = \lim_\omega \frac{d_n(t+s) - 2(r_1(d_n t).r_2(d_n s))_{x_0}}{d_n} t+s,$$

because the sequence of Gromov products $(r_1(d_n t).r_2(d_n s))_{x_0}$ converges to a constant [GH$_1$, Chapitre 7, §3]. The two limit rays, $\rho_1,\ \rho_2$, of the rays $r_1,\ r_2$, defined by $\rho_1(t) = a_t,\ \rho_2(s) = b_s$, have only the origin in common and give therefore distinct directions in $[e]$.

We have found an injective map from $\partial_\infty \mathcal{C}(\Gamma)$ to $\Sigma_{[e]}$.

b) We prove that every direction of $\mathcal{C}(\Gamma)_\omega$ in $[e]$ is determined by a sequence of rays of origin e in $\mathcal{C}(\Gamma)$. A direction of $\mathcal{C}(\Gamma)_\omega$ in $[e]$ is defined by a geodesic segment with one end in $[e]$. This segment is the limit set of a sequence of geodesic segments of $\mathcal{C}(\Gamma)$ with one end in e, of lengths growing linearly in d_n.

We have the following result.

Lemma 3.A.6 ([P$_3$]). *Let (Γ, d_w) be a hyperbolic group endowed with a word metric.*

There exists $M = M(\Gamma) > 0$ such that every geodesic segment sufficiently long in the Cayley graph is contained in the M-neighborhood of a geodesic ray.

Proof. According to [Can] and [ECHPT, Chapter 3, §2], given a hyperbolic group Γ with a set of generators closed under inversion, there exists a finite directed graph $\mathfrak{G}$ with edges labelled by the generators such that every geodesic segment in the Cayley graph of Γ corresponds to a path in $\mathfrak{G}$. If a geodesic segment is long enough, the corresponding path contains at least one loop in $\mathfrak{G}$. The distance of the last loop in this path from its endpoint is bounded by a constant M which depends only on the graph $\mathfrak{G}$. Let ρ be the geodesic ray obtained by going around this loop infinitely many times. The initial segment is contained in $V_M(\rho)$.

$\diamondsuit$

We conclude that every direction of $\mathcal{C}(\Gamma)_\omega$ in $[e]$ is defined by the limit ray of a sequence of rays in $\mathcal{C}(\Gamma)$ of origin e. We then have a surjective map from

$$\mathfrak{R}^{\mathbb{N}} = \{(r_n)_{n\in\mathbb{N}} \mid r_n \text{ geodesic ray in } \mathcal{C}(\Gamma),\ r_n(0) = e\}$$

onto $\Sigma_{[e]}$. It induces a surjective map from $\{(\alpha_n)_{n\in\mathbb{N}} \mid \alpha_n \in \partial_\infty \mathcal{C}(\Gamma)\} = (\partial_\infty \mathcal{C}(\Gamma))^{\mathbb{N}}$ onto $\Sigma_{[e]}$.

Steps a) and b) imply that card $\partial_\infty \mathcal{C}(\Gamma) \leq$ card $\Sigma_{[e]} \leq$ card$(\partial_\infty \mathcal{C}(\Gamma))^{\mathbb{N}}$. For a non-elementary hyperbolic group card $\partial_\infty \mathcal{C}(\Gamma) = 2^{\aleph_0}$. $\diamondsuit$

We now present a result concerning asymptotic subcones of hyperbolic spaces.

Let $\mathcal{S}_1$ be the space of continuous functions $f : [0, r_f] \to \mathbb{R},\ r_f \in [0, \infty)$, such that $f(0) = 0$. For two functions $f_1,\ f_2 \in \mathcal{S}_1$ we define

$$c(f_1, f_2) = \sup\{t \mid f_1(s) = f_2(s),\ \forall s < t\}\,.$$

Then

$$d_1(f_1, f_2) = r_{f_1} + r_{f_2} - 2c(f_1, f_2)\,. \tag{3.1}$$

is a metric on $\mathcal{S}_1$.

Let $\mathcal{S}_2$ be the space of functions $f : [0, r_f) \to \mathbb{R},\ r_f \in (0, \infty)$, such that $f(0) = 0$ and $f(t) \neq 0$ only for t in a finite set. We endow $\mathcal{S}_2$ with a metric d_2 defined as in (3.1).

The spaces $(\mathcal{S}_i,\ d_i),\ i = 1, 2$, are real trees, with every point a branching point for a continuum of directions. The space $(\mathcal{S}_2, d_2)$ can be isometrically embedded into $(\mathcal{S}_1,\ d_1)$, but is not isometric to it. Moreover $\mathcal{S}_1$ is a homogeneous space. It also has the property that every real tree with at most countably many branching points and with order of branching at most $2^{\aleph_0}$ can be isometrically embedded into $\mathcal{S}_1$.

Besides the previous properties of the two spaces, I. Polterovich and A. Schnirelman showed that $\mathcal{S}_1$ is an asymptotic subcone of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ ([PS$_1$], [PS$_2$]). The proof is done by first showing that $\mathcal{S}_2$ is an asymptotic subcone of $\mathbb{H}^2_{\mathbb{R}}$. Since $\mathcal{S}_1$ can be in some sense approximated by $\mathcal{S}_2$, the same follows for $\mathcal{S}_1$. Moreover, the proof for $\mathcal{S}_2$ also implies that any real tree is an asymptotic subcone of $\mathbb{H}^2_{\mathbb{R}}$.

The previous results can be generalized to all complete simply connected manifolds with sectional curvature at most $-k,\ k > 0$ [DP$_2$].

Inspiring themselves from the methods presented previously, A. Dioubina and I. Polterovich ([DP$_1$], [DP$_2$]) have shown, among other things, a stronger result than Proposition 3.A.5. Let $\mathcal{A}$ be the space of functions $f : (-\infty, r_f) \to [0, 1],\ r_f \in (0, \infty)$, such that

• there exists τ_f with the property that $f(t) = 0$ for every $t < \tau_f$;

• for every $t \in (-\infty, r_f)$ there exists $\epsilon > 0$ such that $f|_{[t,t+\epsilon]}$ is constant (f is "piecewise constant from the right").

We endow $\mathcal{A}$ with a metric d defined as in (3.1). The space $(\mathcal{A}, d)$ is an $2^{\aleph_0}$-universal real tree. Also, every complete real tree with all points branching points with branching number $2^{\aleph_0}$ is isometric to $\mathcal{A}$.

Theorem 3.A.7 ([DP$_1$], [DP$_2$]). *(a) Every asymptotic cone of a non-elementary hyperbolic group is isometric to $\mathcal{A}$.*

(b) Every asymptotic cone of a complete, simply connected Riemannian manifold with sectional curvature at most $-k$, $k > 0$ a fixed constant, is isometric to $\mathcal{A}$.

A consequence of Theorem 3.A.7 is that all asymptotic cones of non-elementary hyperbolic groups and of complete, simply connected Riemannian manifolds with strictly negative sectional curvature cannot be distinguished one from the other. It follows that the asymptotic cones as such are an inappropriate tool for any problem in which such a distinction is needed. The question whether asymptotic cones can be endowed with a supplementary structure capable of perceiving the difference between hyperbolic spaces (for instance, between symmetric spaces of rank one) is open.

3.B Applications

A straightforward consequence of the results presented previously is the fact that a weaker sufficient condition for a geodesic metric space to be hyperbolic can be given. First we shall define functions measuring the "size" of geodesic triangles in a geodesic metric space X.

Let $\lambda \in \left(0, \frac{1}{2}\right)$. Given a Lipschitz simple closed curve $c : \mathbb{S}^1 \to X$ of length ℓ, we define the *λ-constriction of the curve c* as $constr_\lambda(c) = \inf\{d(x,y) \mid x, y$ separating $c(\mathbb{S}^1)$ into two arcs of length at least $\lambda\ell\}$. We can define two functions by means of this notion:

- the *λ-constriction function $constr_\lambda$* $: \mathbb{R}_+^* \to \mathbb{R}_+^*$,

$$constr_\lambda(\ell) = \sup\{constr_\lambda(c) \mid c \text{ Lipschitz simple closed curve of length } \leq \ell\}\,;$$

- the *λ-constriction of triangles $Tconstr_\lambda$* $: \mathbb{R}_+^* \to \mathbb{R}_+^*$,

$$Tconstr_\lambda(\ell) = \sup\{constr_\lambda(\Delta) \mid \Delta \text{ geodesic triangle of length } \leq \ell\}\,.$$

For every geodesic triangle Δ, of edges e_1, e_2, e_3, *the minimal size* of Δ is defined as

$$minsize\,\Delta = \inf\{diam\{y_1,\, y_2,\, y_3\} \mid y_i \in e_i,\ i = 1,2,3\}\ .$$

We define the *minimal size function*,

$$minsize : \mathbb{R}_+^* \to \mathbb{R}_+^*,\ minsize(\ell) = \sup\{minsize\,\Delta \mid \Delta \text{ geodesic triangle of length } \leq \ell\}\,.$$

The following weakenings of the "thin triangles condition" are enough for hyperbolicity.

Proposition 3.B.1 ([Gr$_3$], §6, [Dr$_3$], §3). *In a geodesic metric space X the following statements are equivalent:*

(a) X is hyperbolic;

(b) there exists $\lambda \in \left(0, \frac{1}{2}\right)$ such that $Tconstr_\lambda(\ell) = o(\ell)$;

(c) $minsize(\ell) = o(\ell)$.

M. Gromov also proved the following version of the previous statement.

Theorem 3.B.2 (Proposition 6.6.F, [Gr$_3$]). *There exists a universal constant $\varepsilon_0 > 0$ such that if in a geodesic metric space X all geodesic triangles with length $\geq L_0$, for some L_0, have*

$$minsize\,\Delta \leq \varepsilon_0 \cdot length\,\Delta\,,$$

then X is hyperbolic.

The proof with asymptotic cone methods of the previous theorem as well as of a version of Proposition 3.B.1, with condition $Tconstr_\lambda(\ell) = o(\ell)$ replaced by $Tconstr_\lambda(\ell) \leq \varepsilon_0\ell$ with $\varepsilon_0 > 0$ a universal constant, seems beyond reach.

Hyperbolicity also gives information on the general constriction function.

Proposition 3.B.3 ([$\mathbf{Dr_3}$], §3). *In a hyperbolic geodesic metric space X, for all $\lambda \in \left(0, \frac{1}{4}\right]$, $constr_\lambda(\ell) = O(\log \ell)$.*

We note that a better order that $O(\log \ell)$ cannot be obtained for the general constriction function. This can be seen by considering, in the half-space model of $\mathbb{H}^3_{\mathbb{R}}$, the horizontal circle of length ℓ.

Hyperbolic spaces are also interesting from the point of view of the behavior of the filling order. M. Gromov showed the following result, which implies that it always makes sense to speak of the Dehn function in a hyperbolic group.

Theorem 3.B.4 (Theorem 2.3.A, [$\mathbf{Gr_3}$]). *A finitely generated hyperbolic group is always finitely presented.*

As to the filling order of a hyperbolic space, M. Gromov proved it is always linear. Moreover, this characterizes hyperbolicity.

Theorem 3.B.5 (linear filling, §2.3, §6.8**, [$\mathbf{Gr_3}$]).** *A geodesic metric space is hyperbolic if and only if its filling function is linear.*

There is a stronger version of the converse part. This version points out that there is a gap between the quadratic filling order and the linear filling order: as soon as the filling is less than quadratic, it has to be linear and the space has to be hyperbolic.

Theorem 3.B.6 (subquadratic filling, §2.3, §6.8**, [$\mathbf{Gr_3}$]).** *If in a geodesic metric space X the filling function $A_X(\ell) = o(\ell^2)$, then the space is hyperbolic.*

There have been several proofs of Theorem 3.B.6, given by A. Y. Olshanskii [Ol], by P. Papasoglu [P_2] and by B. Bowditch [Bow_1]. B. Bowditch makes use of only two properties of the area function in his proof: the quadrangle inequality and the theta property. By means of asymptotic cone techniques and using only the quadrangle inequality, a very short proof of Theorem 3.B.6 can be given [Dr_3]. Actually, in Theorem 3.B.6 the filling function can be replaced by any function satisfying the quadrangle inequality. We note that in a Riemannian manifold there is another inequality satisfied which can replace the quadrangle inequality: the *inverse isoperimetric inequality* [CDP, Chapter 7, Lemma 1.4]

$$Ar(\Delta) \geq \frac{1}{2\sqrt{3}}[minsize(\Delta)]^2, \ \forall \Delta \text{ geodesic triangle.}$$

The strongest known version of the converse part of Theorem 3.B.5 is as follows.

Theorem 3.B.7 (subquadratic filling-strongest version, §2.3, §6.8**, [$\mathbf{Gr_3}$], [Ol], [$\mathbf{P_1}$]).** *Let X be a geodesic metric space. If there exist N and L big enough and ϵ small enough such that every loop c in X with $N \leq Ar(c) \leq LN$ satisfies*

$$Ar(c) \leq \epsilon [length(c)]^2 ,$$

then the space X is hyperbolic.

It seems impossible to prove with asymptotic cones the previous theorem.

M. Coornaert, T. Delzant and A. Papadopoulos have shown that if X is a complete simply connected Riemannian manifold which is *reasonable* (see [CDP, Chapter6, §1] for a definition of this notion; for instance if X is cocompactly homogeneous, then X is reasonable) then the constant ϵ in the previous theorem only has to be smaller than $\frac{1}{16\pi}$ [CDP, Chapter 6, Theorem 2.1].

As pointed out by M. Gromov in [Gr_3, §6.6], in the two previous theorems only geodesic triangles Δ need to be considered, among closed curves. In the last theorem the condition $N \leq Ar(\Delta) \leq LN$ has to be replaced by *length* $(\Delta) \geq N$.

Remark 3.B.8 ([Ol], [Bat]). *If a finitely presented group Γ has $A(\ell) = o(\ell)$, then Γ is either free or finite.*

Another immediate consequence of the results in Section 3.A is the description of the behavior of the filling radius in hyperbolic spaces.

Proposition 3.B.9 ([$\mathbf{Gr}_3$], §6, [$\mathbf{Dr}_3$], §3). *In a geodesic μ-simply connected metric space X the following statements are equivalent:*

(a) X is hyperbolic;

(b) the filling radius $r(\ell) = o(\ell)$;

(c) the filling radius $r(\ell) = O(\log \ell)$.

The logarithmic order in (c) cannot be improved, as shown by the same example of the horizontal circle in the half-space model of $\mathbb{H}^3_{\mathbb{R}}$. We note that the previous result shows that, as in the case of the filling area, there is a gap between the linear order of the filling radius and the logarithmical one.

Like for the area, for the radius too there is a strongest version of the implication sublinear radius $\Longrightarrow$ hyperbolicity, similar to Theorem 3.B.7.

Proposition 3.B.10 (M. Gromov, [$\mathbf{P}_5$]). *Let Γ be a finitely presented group. If there exists $\ell_0 > 0$ such that*

$$r(\ell) \leq \frac{\ell}{73},\ \forall \ell \geq \ell_0\,,$$

then the group Γ is hyperbolic.

The best possible constant expected is not $\frac{1}{73}$, but $\frac{1}{8}$ [P_5]. Also, there should be a similar statement for general metric spaces, and the constant can probably be specified again for complete simply connected Riemannian manifolds (eventually reasonable).

3.C Dependance of asymptotic cones on the ultrafilter/scalars

We present here an example due to S. Thomas and B. Velickovic [ThV] of a family of finitely generated groups which have two non-homeomorphic asymptotic cones, corresponding to two distinct non-principal ultrafilters or to two distinct sequences of scalars.

All groups have two generators a and b. Their relations are chosen in the sequence of words $w_n(a,b) = (a^n b^n)^7,\ n \in \mathbb{N}$. Let $\mathcal{S} = \{s_1, s_2, \dots, s_m, \dots\}$ be a subset of $\mathbb{N}$ written in an increasing order. To each $\mathcal{S}$ can be associated the group $\Gamma_{\mathcal{S}}$ of presentation $< a, b \mid (w_n(a,b))_{n\in\mathcal{S}} >$. The group $\Gamma_{\mathcal{S}}$ satisfies the $C'(1/6)$-cancellation property. This implies the following

Lemma 3.C.1. *Every word w in a and b such that $w \equiv e$ in $\Gamma_{\mathcal{S}}$ can be written as $w = \prod_{i=1}^{N} \gamma_i r_i \gamma_i^{-1}$, where γ_i are in the free group generated by a and b and r_i are in the normal closure of the set $\{w_n(a,b) \mid n \in \mathcal{S},\ n < \frac{1}{7} \text{length } (w)\}$.*

The group Γ_m of presentation $< a, b \mid (w_n(a,b))_{n\in\{s_1, s_2, \dots, s_m\}} >$ also has the $C'(1/6)$-cancellation property and, being a finitely presented group, it is hyperbolic. Let δ_m be its constant of hyperbolicity.

S. Thomas and B. Velickovic show that if the set $\mathcal{S} = \{s_1, s_2, \dots, s_m, \dots\}$ is chosen sufficiently sparse, then at least one asymptotic cone of $\Gamma_{\mathcal{S}}$ is a real tree. Namely, it is enough if there exists an increasing sequence (t_m) such that $s_m < t_m < s_{m+1}$, $\frac{\delta_m}{t_m} \to 0$ and $\frac{s_{m+1}}{t_m} \to \infty$. By Remark 2.A.1 there exists a nonprincipal ultrafilter ω such that $\omega(\{t_1, t_2, \dots, t_m, \dots\}) = 1$. Then the following is true.

Proposition 3.C.2 ([ThV]). *The asymptotic cone $(\Gamma_{\mathcal{S}})_\omega((e),(n))$ is a real tree.*

An outline of the argument proving this is as follows. Every ball $B([e], R)$ in the previous asymptotic cone is more or less the ω-limit of the sequence of balls $B(e, Rn)$ in the Cayley graph of $\Gamma_{\mathcal{S}}$. Since $\omega(\{t_1, t_2, \dots, t_m, \dots\}) = 1$, it is enough to look at the sequence of balls $B(e, Rt_n)$. By Lemma 3.C.1 the behavior of geodesic triangles in the ball $B(e, Rt_n)$ in the Cayley graph of $\Gamma_{\mathcal{S}}$ is the same as in the graph of Γ_m. This and the fact that $\frac{\delta_m}{t_m} \to 0$ imply that $B([e], R)$ is isometric to a ball in a real tree. Since this works for all $R > 0$ the conclusion follows.

If, for ω fixed, all the asymptotic cones $(\Gamma_{\mathcal{S}})_\omega((e),(d_n))$ would be real trees, then, by Proposition 3.A.1 and Remark 2.A.2, the group $\Gamma_{\mathcal{S}}$ would be hyperbolic. Likewise, if all the asymptotic cones $(\Gamma_{\mathcal{S}})_\nu((e),(n))$, with ν arbitrary, would be real trees, then $\Gamma_{\mathcal{S}}$ would be hyperbolic. By Theorem 3.B.4, $\Gamma_{\mathcal{S}}$ would then be finitely presented. Since this is impossible, it follows that there exists a non-principal ultrafilter ν such that $(\Gamma_{\mathcal{S}})_\nu((e),(n))$ is not a real tree, and that for the chosen non-principal ultrafilter ω there exists a sequence of scalars (d_n) such that $(\Gamma_{\mathcal{S}})_\omega((e),(d_n))$ is not a real tree. In particular the previous two asymptotic cones are not homeomorphic to the asymptotic cone in Proposition 3.C.2, since real trees admit a topological characterization in the class of geodesic metric spaces.

In the example of S. Thomas and B. Velickovic discussed previously, the fact that the group is not finitely presented plays an important part. The question whether for finitely presented groups two asymptotic cones can be non-homeomorphic (non-isometric) is open.

4 Polynomial filling order and asymptotic cones

We study the relationship between the filling order of a space and properties of its asymptotic cones. We have, first of all, the following result.

Theorem 4.1 (M. Gromov, [Gr$_3$], §5.F). *Let (X,d) be a geodesic metric space. If for every non-principal ultrafilter ω, every sequence of observation centers (x_n) and every sequence of scalars (d_n), $X_\omega((x_n),(d_n))$ is simply connected then:*

(i) *The space (X,d) is μ-simply connected. Let $A^X(\ell)$ be a filling function of X.*

(ii) *There exist $a \geq 0, \alpha \geq 1$ and $\ell_0 > 0$ such that*
$$A^X(\ell) \leq a\ell^\alpha \text{ for } \ell \geq \ell_0.$$

M. Gromov conjectured in [Gr$_4$] that there should also be a relation between the order of the higher dimensional filling functions and the higher homotopy groups of the asymptotic cones, similar to the one in the previous theorem.

In the proof of Theorem 4.1 there is an essential intermediate step.

Definition 4.2. *We say that a geodesic metric space X satisfies the Loop Division Property if there exist $\nu_0 \in \mathbb{N}$, $M \in \mathbb{N}$, $M \geq 2$, and $\ell_0 > 0$ such that*

$$P(\ell, \ell/M) \leq \nu_0,\ \forall \ell \geq \ell_0\,.$$

The Loop Division Property ensures a polynomial filling order, as shown in the sequel.

Lemma 4.3. *If a geodesic metric space X satisfies the Loop Division Property for some $\nu_0 \in \mathbb{N}$, $M \in \mathbb{N}$, $M \geq 2$, and $\ell_0 > 0$ then:*

(1) *The space (X,d) is μ-simply connected. Let $A^X(\ell)$ be a filling function of X.*

(2) *We have $A^X(\ell) \leq a\ell^\alpha$ for $\ell \geq \ell_0$, where $\alpha = \frac{\ln \nu_0}{\ln M}$ and a is a constant.*

Proof of the Lemma. Let c be a Lipschitz closed curve of length at most ℓ. There exists a partition of c into ν_0 bricks of mesh at most $\frac{\ell}{M}$. If the mesh is smaller than ℓ_0 we stop. If not, for every brick in the partition we consider a partition into ν_0 bricks. We obtain a partition of c into ν_0^2 bricks of mesh at most $\frac{\ell}{M^2}$. If the mesh is smaller than ℓ_0 we stop, otherwise we continue. In the worst case we shall stop after p steps, where $p \in \mathbb{N}$ is minimal such that $\frac{\ell}{M^p} \leq \ell_0 \Longleftrightarrow p \geq \log_M \frac{\ell}{\ell_0}$. It follows $p = \left[\log_M \frac{\ell}{\ell_0}\right] + 1$.
Then
$$P(\ell, \ell_0) \leq \nu_0^p \leq a\nu_0^{\log_M \ell} = a\ell^\alpha, \alpha = \frac{\ln \nu_0}{\ln M}\,.$$

Thus, X is ℓ_0-simply connected. Obviously $A^X(\ell) \doteq P(\ell, \ell_0) \prec \ell^\alpha$.

$\diamondsuit$

The following equivalence suffices therefore to conclude the proof of Gromov's Theorem.

Theorem 4.4 (M. Gromov, P. Papasoglu, [Gr$_4$], [P$_4$]). *Let X be a geodesic metric space. All its asymptotic cones are simply connected if and only if X satisfies the Loop Division Property.*

Proof. We first prove that the simple connectedness of asymptotic cones implies the Loop Division Property. We argue by contradiction and we suppose the contrary:

$$\forall M,\ \forall \nu_0,\ \forall \ell_0,\ \exists \ell \geq \ell_0 \text{ such that } P(\ell, \ell/M) > \nu_0.$$

We fix $M \in \mathbb{N}, M \geq 2$, and we choose sequences ν_n and ℓ'_n diverging to $+\infty$. We obtain the existence of a sequence of closed Lipschitz curves c_n of length $\ell_n \geq \ell'_n$ such that $\mathrm{Mesh}(c_n, \nu) > \ell_n/M, \forall \nu \leq \nu_n$. We may suppose that the curves c_n are parameterized proportionally to the length.

If we choose the observation centers $x_n \in c_n(\mathbb{S}^1)$ and the sequence of scalars (ℓ_n), in $X_\omega((x_n),(\ell_n))$ the sequence of curves (c_n) gives a Lipschitz curve $c : \mathbb{S}^1 \to X_\omega$ of length 1. The curve c has the property that for all $\nu \in \mathbb{N}$, $\mathrm{Mesh}(c, \nu) \geq 1/M$, because each partition of c, $\sigma : \wp \to X_\omega$, comes from a sequence of partitions of c_n, $\sigma_n : \wp \to X$, and we have

$$\mathrm{Mesh}(\sigma) = \lim_\omega \frac{\mathrm{Mesh}(\sigma_n)}{\ell_n} \geq \frac{1}{M}.$$

On the other hand, by hypothesis, c is contractible. Therefore there exists $\sigma : \mathbb{D}^2 \to X_\omega$, $\sigma|_{\mathbb{S}_1} = c$, σ continuous and so uniformly continuous. For every $\varepsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in \mathbb{D}^2,\ d(x,y) < \delta \Rightarrow d(\sigma(x), \sigma(y)) < \varepsilon$. We choose $\varepsilon < \frac{1}{3M}$.

If we consider a net of $\mathbb{D}^2$ composed of triangles of edge δ, the restriction of σ to the vertices of this net is a partition of c with Mesh $< 3\varepsilon < \frac{1}{M}$. We have obtained a contradiction.

We only give an outline of the proof of the converse part. It has been proven rigorously in [P_4], following an idea of M. Gromov [Gr_4]. The Loop Division Property is inherited by all asymptotic cones. By applying it an infinite number of times in a fixed cone, for a fixed loop, and by using the completeness of the cone, a topological disk filling the loop is obtained. $\diamondsuit$

Remark 4.5. *We have the following sequence of implications: every asymptotic cone of a space X is simply connected $\Rightarrow$ X satisfies the Loop Division Property, for some fixed $M \geq 2$, ν_0 and ℓ_0 $\Rightarrow$ every asymptotic cone satisfies the Loop Division Property, for the same M, $\nu'_0 = \nu_0 + 2M$, $\ell_0 = 0$ $\Rightarrow$ every asymptotic cone has a polynomial filling order of degree at most $\alpha = \frac{\ln(\nu_0+2M)}{\ln(M)}$.*

Proof. We only have to prove the second implication. For every Lipschitz closed curve $c : \mathbb{S}^1 \to X_\omega$, of length at most ℓ, we can choose $2M$ points on $c(\mathbb{S}^1)$, P_1, P_2, …,P_{2M}, which split $c(\mathbb{S}^1)$ into $2M$ arcs of equal length. We join each P_i to P_{i+1} by a geodesic arc. We obtain a polygon, which is the limit of a sequence of polygons $P_1^{(n)} - P_2^{(n)} - \dots P_{2M}^{(n)}$, where $P_i = [P_i^{(n)}]$. For each polygon of the sequence there exists a partition into ν_0 bricks of mesh smaller than the length of the polygon $\times \frac{1}{M}$. We note that, combinatorially speaking, there is a finite set T of possible triangulations of $\mathbb{D}^2$ with less than ν_0 triangles. For every $t \in T$ we denote by A_t the set of $n \in \mathbb{N}$ such that the triangulation in the partition of the polygon $P_1^{(n)} - P_2^{(n)} - \dots P_{2M}^{(n)}$ is t. The fact that $\mathbb{N} = \sqcup_{t \in T} A_t$ and the fact that ω is finitely additive and can only take values 0 and 1 imply that $\omega(A_{t_0}) = 1$ for some t_0 and $\omega(A_t) = 0$ for all $t \neq t_0$. Thus, we may suppose that all the triangulations of the partitions in our sequence are identical. The sequence of partitions will give a partition of the polygon $P_1 - P_2 - \dots - P_{2M}$ whose associated triangulation of $\mathbb{D}^2$ will be the same, which will have at most ν_0 bricks and a mesh $\leq \frac{\ell}{M}$.

Thus we have obtained a partition of c into at most $\nu_0' = \nu_0 + 2M$ bricks, with mesh $\leq \frac{\ell}{M}$. $\diamondsuit$

We mention here a theorem of P. Papasoglu giving, in some sense, a more precise description of the relation between the filling in the asymptotic cones and the filling in the space.

Theorem 4.6 (P. Papasoglu, [P$_3$], [Dr$_2$]). *Let X be a geodesic metric space. If in every asymptotic cone of X we have that*

$$A(\ell) \leq C \cdot \ell^p\,,\ \textit{for all } \ell,$$

where C is an universal constant, then in the space X we have that for every $\varepsilon > 0$ there exists ℓ_ε such that

$$A^X(\ell) \leq \ell^{p+\varepsilon}\,,\ \textit{for all } \ell \geq \ell_\varepsilon\,.$$

The proof of Theorem 4.6 follows the same lines as the proof of Gromov's Theorem. It is shown that the hypothesis implies that X satisfies the Loop Division Property for every $M \in \mathbb{N},\ M \geq 2$, for $\nu_0 = \varkappa M^p$, where $\varkappa$ is an universal constant, and for $\ell_0 = \ell_0(M)$. In order to obtain the conclusion it suffices now to apply Lemma 4.3 for every $M \in \mathbb{N},\ M \geq 2$, for $\nu_0 = \varkappa M^p$ and for $\ell_0 = \ell_0(M)$.

In [P$_4$], P. Papasoglu proves a kind of reciprocal of Theorem 4.1.

Theorem 4.7 ([P$_4$]). *If a finitely presented group has a quadratic filling order then all its asymptotic cones are simply connected.*

According to Theorem 4.4, the proof of the previous theorem reduces to proving that a quadratic filling order implies the Loop Division Property. This is shown in [P$_4$], for some $\nu_0 \in \mathbb{N}$ and $M = 2$.

Concerning the filling radius, the following result holds.

Theorem 4.8 ([P$_4$]). *If all asymptotic cones of a finitely presented group are simply connected then the filling radius grows linearly.*

The reciprocal of Theorem 4.1 in full generality is not true. This is because there exist finitely presented groups with polynomial filling order and filling radius growing faster than linear. The existence of such groups follows from Corollary 2.B.3. In [Br$_1$], other examples are provided.

The question whether polynomial filling order and linear filling radius imply that all asymptotic cones are simply connected remains open.

5 Polynomial growth and asymptotic cones: nilpotent groups

The result relating the order of the growth function to the topological properties of the asymptotic cones is the following.

Theorem 5.1. *Let (Γ, d_w) be a finitely generated group with a word metric.*

a) $\Gamma_\omega((e),(d_n))$ *is a proper metric space for all* ω *and* $(d_n) \iff (\Gamma, d_w)$ *has polynomial growth* $\iff \Gamma$ *is virtually nilpotent.*

Suppose in the sequel that Γ *is virtually nilpotent. Let* $tor(\Gamma)$ *be the finite normal subgroup of* Γ *generated by its torsion elements and let* N *be the torsion free nilpotent group* $\Gamma/tor(\Gamma)$. *Let* $N^1 = N$, $N^{i+1} = [N, N^i]$, $\forall i \geq 1$, *be the groups of the lower central series of* N *and let* r_i *denote* $rank(N^i/N^{i+1})$.

b) *All the asymptotic cones are isometric, and isometric to a graded Lie group* G_∞ *endowed with a Carnot-Caratheodory metric (see [Pa$_1$] for a definition of the group* G_∞ *and of its metric);* $(\Gamma, \varepsilon \cdot d_w)$ *converges in the Hausdorff modified metric, for* $\varepsilon \to 0$, *to this graded Lie group.*

c) *The dimension of the limit space is equal to the cohomological dimension of* N *and to* $\Sigma_i r_i$.

d) *The Hausdorff dimension of the limit space is equal to the degree of polynomial growth and to* $\Sigma_i i\, r_i$.

The second equivalence in statement a) of the previous theorem is the celebrated Polynomial Growth Theorem of M. Gromov ([Gr$_1$], [VDW]), and it implies that virtual nilpotency is a geometric property in the class of finitely generated groups. We should mention here that although virtual nilpotency is one of the first properties known to be geometric, as far as the classification up to quasi-isometry of nilpotent groups is concerned, almost nothing is known.

The second equality of d) is an earlier result of H. Bass [Bas].

Statement d) emphasizes that the degree of polynomial growth, if it is not infinite, it is a quasi-isometry invariant. See [GH$_2$] for a definition and properties of the degree of polynomial growth.

It has been shown by S. Gersten ([Ge$_2$], see also [BW]) that in the class of groups with finite classifying space (which includes the class of nilpotent torsion free groups) the cohomological dimension is a quasi-isometry invariant. The previous result also implies that the cohomological dimension is a quasi-isometry invariant in the class of nilpotent torsion free groups, moreover providing a new geometric significance for it.

The quasi-isometry invariance of both the degree of polynomial growth and the cohomological dimension implies that a group quasi-isometric to $\mathbb{Z}^n$ has a finite index subgroup isomorphic to $\mathbb{Z}^n$, as it has been shown in [BG]. The question of finding an elementary proof of this, not involving Gromov's Polynomial Growth Theorem, asked in [GH$_1$], is still open.

The statement b) as well as the first equalities in c) and d) are proved by P. Pansu [Pa_1]. Concerning the behavior of the limit space up to quasi-isometry he proves a stronger result then just the bi-Lipschitz equivalence; if two virtually nilpotent groups are quasi-isometric then their limit graded groups are isomorphic [Pa_2]. This in particular implies that the degree of nilpotency of N as well as each of the numbers r_i is a quasi-isometry invariant.

It is proved in [KaL_1] that if $(\Gamma, \varepsilon \cdot d_w)$ converges in the Hausdorff modified metric, for $\varepsilon \to 0$, to a limit space, then all the asymptotic cones are isometric to the limit space.

So the only thing that remains to be proved is the direct part of statement a).

Proof of the direct part of statement a). Suppose $\Gamma_\omega((e),(d_n))$ is proper for all ω and (d_n). We show there exist $C > 0$, $r_0 > 0$, $d > 0$ such that card $B(e,r) \leq C \cdot r^d, \forall r \geq r_0$.

We argue by contradiction: suppose that for all $n \in \mathbb{N}$ there exists $R_n \geq 1$ such that

$$\text{card } B(e, R_n) \geq n \cdot (R_n)^n \,.$$

For every $n \in \mathbb{N}$ we choose the R_n minimal satisfying this relation. Obviously $R_n \to \infty$.

We consider $\Gamma_\omega((e),(R_n))$. Let $\lambda \in (0,1)$. As the ball $B([e],2)$ is compact, it has a finite covering with balls of radius λ,

$$B([e],2) \subset \bigcup_{i=1}^{k} B([x_n^i],\lambda)\,.$$

It follows, with the notations in the end of Section 2.A, that

$$\pi^{-1}(B([e],2)) \subset \bigcup_{i=1}^{k} \pi^{-1}(B([x_n^i],\lambda)). \tag{5.1}$$

By definition, $\pi^{-1}(B([x_n],R)) = \{[[y_n]] \mid \lim_\omega \frac{d(y_n,x_n)}{R_n} < R\}$. We have the sequence of equivalences:

$$\begin{aligned} \lim\nolimits_\omega \tfrac{d(y_n,x_n)}{R_n} < R \quad &\Leftrightarrow \exists \varepsilon > 0 \text{ s.t. } d(y_n,x_n) < (R-\varepsilon)\cdot R_n \ \ \omega - \text{a.s.} \\ &\Leftrightarrow \exists \varepsilon > 0 \text{ s.t. } y_n \in B(x_n,(R-\varepsilon)\cdot R_n) \ \ \omega - \text{a.s.} \\ &\Leftrightarrow \exists \varepsilon > 0 \text{ s.t. } [[y_n]] \in [[B(x_n,(R-\varepsilon)R_n)]] \\ &\Leftrightarrow [[y_n]] \in \bigcup_{0<\varepsilon<R} [[B(x_n,(R-\varepsilon)R_n)]]. \end{aligned}$$

It follows that $\pi^{-1}(B([x_n],R)) = \bigcup_{0<\varepsilon<R} [[B(x_n,(R-\varepsilon)R_n)]]$.

The relation (5.1) may then be written as follows

$$\bigcup_{0<\varepsilon<2} [[B(e,(2-\varepsilon)\cdot R_n)]] \subset \bigcup_{i=1}^{k} \bigcup_{0<\varepsilon_i<\lambda} [[B(x_n^i,(\lambda-\varepsilon_i)R_n)]]. \tag{5.2}$$

We choose $\lambda' \in (\lambda,1)$. The relation (5.2) implies

$$[[B(e,R_n)]] \subset \bigcup_{i=1}^{k} [[B(x_n^i,\lambda' R_n)]]\,. \tag{5.3}$$

We now use the theorem of Los ([VDW], p.361):

Theorem 5.2. *Let $W_1 = [[W_{1,n}]], \dots W_m = [[W_{m,n}]]$ be internal sets of X^* and $f_1, \dots f_k \in X^*$, and let ϕ be a property which can be expressed using $W_1, \dots W_m$, $f_1, \dots f_k$ and the relations $\in$, $\subset$ and the logical quantifiers. Then $W_1, \dots W_m, f_1 \dots f_k$ satisfy property ϕ if and only if $W_{1,n} \dots W_{m,n}, f_1(n), \dots f_k(n)$ satisfy ϕ ω-almost surely.*

According to Los' Theorem, (5.3) is equivalent to

$$B(e, R_n) \subset \bigcup_{i=1}^{k} B(x_n^i, \lambda' R_n) \ \omega\text{−a.s.}$$

It follows card $B(e, R_n) \leq k$ card $B(e, \lambda' R_n)$ ω-a.s.

Then the following sequence of inequalities is true ω-almost surely:

$$n \cdot (R_n)^n \leq \text{card } B(e, R_n) \leq k \cdot \text{card } B(e, \lambda' R_n) \leq k \cdot n \cdot (\lambda' R_n)^n.$$

This implies $(\lambda')^n \geq \frac{1}{k}$ ω−a.s. Since $\lambda' \in (0, 1)$, we have obtained a contradiction.

$\diamondsuit$

The graded Lie groups G_∞ which appear as limits in the Hausdorff modified metric of virtually nilpotent groups are simply connected. This and Theorems 4.1 and 4.8 imply that virtually nilpotent groups have polynomial Dehn functions and linear filling radius. M. Gromov stated the following result on the filling order of a nilpotent group.

Theorem 5.3 ([Gr$_4$], §5.*A***).** *The filling order of a nilpotent group of class k is at most polynomial of degree $k + 1$.*

In [GHR] a detailed and complete proof of Theorem 5.3 can be found. A proof of Theorem 5.3 in some particular cases had been given previously in [Pi$_1$].

The previous bound is sharp, as shown by the following result.

Theorem 5.4 ([BMS], [Gr$_4$]). *Let Γ be a nilpotent group of class k and let Γ^k be the k-th term of its lower central series.*

If Γ^k contains an infinite cyclic subgroup, then the quotient group Γ/Γ^k has a polynomial filling order of degree k.

Theorem 5.4 applies for instance to free nilpotent groups of class k with at least 2 generators, which appear as quotients of free nilpotent groups of class $k + 1$ by the $(k + 1)$-th term of their lower central series.

A similar estimate holds for the abelian-by-cyclic nilpotent groups.

Theorem 5.5 ([Pi$_2$], [BP], [BMS]). *The filling order of a nilpotent group of class k of the form $\mathbb{Z}^n \rtimes_M \mathbb{Z}$, where $M \in GL(n, \mathbb{Z})$, is polynomial of degree $k + 1$.*

For a discussion of the general case of abelian-by-cyclic groups see Section 6.

The filling order in the Heisenberg groups is as follows.

Theorem 5.6 ([ECHPT], [Gr$_4$], [All]). *(a) The filling order in $\mathcal{H}^3$ is cubic.*

(b) The filling order in $\mathcal{H}^{2n+1}$, $n \geq 2$, is quadratic.

The first statement was proved by Thurston in [ECHPT] and used to deduce that $\mathcal{H}^3$ is not automatic. See also Corollary 6.6 in the next section. A definition of automatic and biautomatic groups as well as a presentation of their properties can be found in [ECHPT]. One of the important properties of automatic groups is that they have quadratic filling order.

The second statement was conjectured by Thurston in [ECHPT], Gromov gave an outline of a proof in [Gr_4] and D. Allcock gave the complete proof [All] by means of symplectic geometry. Y. Olshanskii and M. Sapir later gave a combinatorial proof [OS].

6 Solvable groups

We begin with the following dichotomy due to J. A. Wolf and to J. Milnor.

Theorem 6.1 ([Wo], [Mi]). *A finitely generated solvable group either has polynomial growth and is virtually nilpotent or has exponential growth.*

In particular this theorem says that solvable non-virtually nilpotent groups will have non-proper asymptotic cones.

A. Dioubina showed [Di] that the property of being virtually solvable is not a geometric property in the class of groups anymore. Moreover, the examples of A. Dioubina are finitely generated abelian-by-cyclic groups, and they are shown to be quasi-isometric to groups which are not virtually solvable. Thus, the class of finitely generated abelian-by-cyclic groups is not closed under quasi-isometry either. An *abelian-by-cyclic* group is a group Γ such that there exists an exact sequence

$$1 \to A \to \Gamma \to C \to 1\,,$$

where A is abelian and C is infinite cyclic.

On the other hand, B. Farb and L. Mosher have shown that the class of finitely presented nonpolycyclic abelian-by-cyclic groups is closed under quasi-isometry. We recall that a polycyclic group is a group which has a finite subnormal series with cyclic quotients (finite or infinite). The *Hirsch number* of a polycyclic group is the number of infinite cyclic quotients in such a subnormal series.

The very first step in the argument of B. Farb and L. Mosher is to consider torsion free finite index subgroups and to apply a result of R. Bieri and R. Strebel [BS_1], stating that for every torsion-free finitely presented abelian-by-cyclic group there exists $n \in \mathbb{N}$ and a matrix $M = (m_{ij}) \in M(n, \mathbb{Z})$ with non-zero determinant such that the group has the presentation

$$< a_1,\, a_2,\, \ldots,\, a_n,\, t \mid [a_i, a_j] = 1, t a_i t^{-1} a_1^{m_{1i}}\, a_2^{m_{2i}}\, \cdots\, a_n^{m_{ni}} > \ .$$

We denote by Γ_M the group with the presentation determined by the integral matrix M. The group Γ_M is polycyclic if and only if $|det\ M| = 1$ [BS_2].

The following result of closure under quasi-isometry holds.

Theorem 6.2 ([FM$_3$]). *Let M be an integer matrix with $|det\ M| > 1$. If a finitely generated group G is quasi-isometric to the group Γ_M then there exists a finite normal subgroup F of G and an integer matrix N with $|det\ N| > 1$ such that G/F is abstractly commensurable to the group Γ_N.*

Two groups are said to be abstractly commensurable if they have isomorphic finite index subgroups. The previous result is completed by a rigidity result.

Theorem 6.3 ([FM$_3$]). *Let M_1 and M_2 be integer matrices with $|det\ M_i| > 1$, $i = 1, 2$. The groups Γ_{M_1} and Γ_{M_2} are quasi-isometric if and only if there exist two positive integers k_1 and k_2 such that $M_1^{k_1}$ and $M_2^{k_2}$ have the same absolute Jordan form.*

The absolute Jordan form of a matrix is obtained from the Jordan form over $\mathbb{C}$ by replacing the diagonal entries with their absolute values and arranging the Jordan blocks in a canonical way.

In the case of the Baumslag-Solitar solvable groups, which form a subclass in the previous class, two groups are quasi-isometric if and only if they are abstractly commensurable ([FM$_1$], [FM$_2$]).

The question of the closure with respect to quasi-isometry remains open for finitely presented polycyclic abelian-by-cyclic groups, as well as for other subclasses of solvable groups, as the larger class of polycyclic groups, that of metabelian groups (i. e. solvable groups of length 2), etc.

For a nice survey of the problems of classification and of the geometric properties in the class of finitely generated solvable groups see [FM$_4$].

M. Bridson and S. Gersten have shown [BG] that the Hirsch number is a quasi-isometric invariant in the class of polycyclic groups. They have also shown that two groups $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ and $\mathbb{Z}^m \rtimes_{M'} \mathbb{Z}$, with $M \in GL(n, \mathbb{Z})$ and $M' \in GL(m, \mathbb{Z})$ unipotent matrices, are quasi-isometric if and only if $n = m$ and M, M' have the same canonical Jordan form. An important ingredient in the proof of the last result is the previously cited result of P. Pansu, on the quasi-isometry invariance in the class of nilpotent groups of the length of the lower central series as well as of the numbers $rank(N^i/N^{i+1})$.

We shall now discuss several results concerning the filling order for solvable groups. First we must mention an example due to O. Kharlampovič of a finitely presented solvable group of length 3 with non-recursive Dehn function. The length 3 cannot be improved to length 2, as metabelian groups always have recursive Dehn function (see [Kh$_1$] and references therein). In [KhS] can be found a survey of results on the solvability and non-solvability of the word problem in different classes of solvable groups.

As far as finitely presented polycyclic abelian-by-cyclic groups are concerned, the following results are known. W. Thurston has shown that the group Sol has exponential filling, as well as all its uniform lattices, concluding that all these lattices are not automatic [ECHPT]. M. Bridson and S. Gersten have proved the following.

Theorem 6.4 ([BG]). *The filling order of a group $\mathbb{Z}^n \rtimes_M \mathbb{Z}$, where $M \in GL(n, \mathbb{Z})$, is either polynomial or exponential. It is polynomial if and only if all of the eigenvalues of M are roots of unity; the degree of the polynomial is then $d + 1$, where d is the size of the largest elementary block in the Jordan form of M.*

The proof of the previous result relies on the following theorem and on the study of the behavior of $\|M^k\|$.

Theorem 6.5. *Any filling function of a group $\mathbb{Z}^n \rtimes_M \mathbb{Z}$, where $M \in GL(n, \mathbb{Z})$, is equivalent to the function $f : \mathbb{N} \to \mathbb{N},\ f(k) = k^2\|M^k\|$.*

We mention that the relation $A \prec k^2\|M^k\|$, where A denotes a filling function of the group, had been proven already by M. Bridson and C. Pittet in [BP]. In the same paper they also proved Theorem 6.5 for n even and $M \in Sp(n/2, \mathbb{Z})$.

Besides being the main ingredient in the proof of Theorem 6.4, Theorem 6.5 has two immediate consequences.

Corollary 6.6 ([BP]). *The filling order in Nil is cubic and in Sol it is exponential.*

To deduce the first statement, it suffices to see that $\mathbb{Z}^2 \rtimes_U \mathbb{Z}$, with

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

is a cocompact lattice in Nil and that $\|U^k\|$ grows linearly.

The second statement follows from the fact that $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$, with

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

is a cocompact lattice in Sol and the fact that $\|M^k\|$ grows exponentially (M has a real eigenvalue bigger than 1).

Corollary 6.7. *Let Γ denote $\mathbb{Z}^n \rtimes_M \mathbb{Z}$, where $M \in GL(n, \mathbb{Z})$. The following statements are equivalent:*

(a) Γ satisfies a quadratic isoperimetric inequality;

(b) M has finite order;

(c) Γ is virtually abelian;

(d) Γ is automatic;

(e) Γ is biautomatic.

The Baumslag-Solitar groups $BS(p, p)$ are automatic, so they have a quadratic filling order. On the other hand, S. Gersten showed that the Baumslag-Solitar groups $BS(p, q)$ with $p \neq q$ have exponential filling order [Ge$_3$], which in particular implies that they are not automatic, while they are known to be asynchronously automatic. We recall that a Baumslag-Solitar group is solvable if $p = 1$. The result of S. Gersten agrees with the known properties of the asymptotic cones of the Baumslag-Solitar groups. M. Gromov has shown [Gr$_4$] that their asymptotic cones, as well as those of the group Sol, are not simply connected. J. Burillo has shown moreover the following.

Theorem 6.8 ([Bu]). *The fundamental group of every asymptotic cone of the Baumslag-Solitar group $BS(p,q)$ is uncountable and non-free. The same is true for the group Sol.*

The previous theorem is proved by exhibiting a subspace in each asymptotic cone which is homotopically equivalent to the Hawaiian earring.

In [Gr_4, §5.A_9] M. Gromov stated that $\mathbb{R}^n \rtimes \mathbb{R}^{n-1}$, $n \geq 3$, has quadratic filling order, and gave an idea of the proof. In fact for the previous group, as well as for a large class of Lie groups, a geometric model can be provided and by means of it the quadratic filling order obtained.

First we recall the definition of horospheres in nonpositively curved spaces. Let X be a CAT(0)-space (see [BH] for a definition) and let ρ be a geodesic ray in X. *The Busemann function associated to* ρ is the function $f_\rho : X \to \mathbb{R}$, $f_\rho(x) = \lim_{t\to\infty}[d(x, \rho(t)) - t]$. This function is well defined and convex. Its level hypersurfaces $H_a(\rho) = \{x \in X \mid f_\rho(x) = a\}$ are called *horospheres* .

For two asymptotic rays, their Busemann functions differ by a constant [BH]. Thus, the families of horospheres are the same, and we shall call them horospheres *of basepoint* α, where α is the common point at infinity of the two rays.

If X is a symmetric space, for every geodesic ray ρ, there exists a solvable subgroup P_0 of $Isom_0(X)$ which stabilizes all horospheres $H_a(\rho)$ and acts freely transitively on each of them. If ρ is not parallel to a rank 1 factor of X, any left invariant metric on P_0 is bi-Lipschitz equivalent to the induced metric [Dr_1].

Examples of solvable groups in the previous class:

(1) All solvable groups $Sol_{2n-1}(\phi) = \mathbb{R}^n \rtimes_\phi \mathbb{R}^{n-1}$, where $n \geq 2$ and $\phi : \mathbb{R}^{n-1} \to Gl(n, \mathbb{R})$ is an injective homomorphism with diagonalizable image, act freely transitively on horospheres of $\mathbb{H}^2 \times \mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ (n times).

(2) The groups of upper (lower) triangular matrices of order n with the set of diagonals of the form $\{(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \mid \sum_{i=1}^n \lambda_i = 0, \sum_{i=1}^n u_i\lambda_i = 0\}$ for a fixed vector $(u_1, u_2, \ldots, u_n) \neq (1, 1, \ldots, 1)$, act freely transitively on horospheres of $SL(n, \mathbb{R})/SO(n)$.

For all the groups acting on horospheres we have the following result.

Theorem 6.9 ([Dr_4]). *Let X be a symmetric space of rank at least* 3*. Let ρ be a geodesic ray which is not contained in a rank one or in a rank two factor of X.*

(1) The Riemannian filling order in every horosphere $H_a(\rho)$ of X is quadratic.

(2) Let P_0 be a Lie group acting by isometries, transitively with compact stabilizers on a horosphere $H_a(\rho)$ of X. In P_0 endowed with any left invariant metric the filling order is quadratic. The same is true for every discrete group Γ acting properly discontinuously cocompactly on $H_a(\rho)$, Γ endowed with a word metric.

Statement (1) remains true if X is a product of symmetric spaces and Euclidean buildings of rank at least 3.

The quadratic estimate on the filling function is sharp. This is because in a symmetric space of rank $r \geq 3$, every horosphere contains isometric copies of the Euclidean space $\mathbb{E}^{r-1}$ and there are canonical projections of the horosphere on each of these copies decreasing the distance.

As to the existence of solvable discrete groups acting properly discontinuously cocompactly on horospheres, we shall recall some known facts in the sequel. Since we are interested in solvable groups, we shall discuss mainly the existence of cocompact lattices in the solvable stabilizers of horospheres. We note that such a group, whenever it exists, is polycyclic, as lattice in a Lie group.

Let Γ be a lattice in the solvable stabilizer P_0 of a horosphere $H_a(\rho)$. The existence of Γ implies that P_0 is unimodular. On the other hand, P_0 is unimodular if and only if the corresponding geodesic ray ρ has a barycentric direction in the Weyl chamber containing it. To see this it suffices to consider the decomposition of P_0 as $P_0 = A^{\perp}N$, where N and A are respectively the maximal unipotent and the maximal $\mathbb{R}$-diagonalizable torus corresponding to ρ and $A^{\perp}$ is the codimension 1 torus in A orthogonal to the direction corresponding to ρ. It can be shown that for every $g \in P_0,\ g = an,\ a \in A^{\perp},\ n \in N$, we have $\det Ad(g) = \det Ad(a) = e^{\Sigma_{\alpha\in\Pi_+}\alpha(X)}$, where X is the vector in the Lie algebra $\mathfrak{a}^{\perp}$ of $A^{\perp}$ such that $exp(X) = a$, and Π_+ is the set of roots which are positive on a Weyl chamber containing the direction corresponding to ρ. The group P_0 is unimodular if and only if $\det Ad(g) = 1$ for all $g \in P_0$, so if and only if $\Sigma_{\alpha\in\Pi_+}\alpha(X) = 0$ for all $X \in \mathfrak{a}^{\perp}$. The last equality is equivalent to the fact that ρ has a barycentric direction.

Thus, the only horospheres on which a solvable discrete group might act freely and properly discontinuously cocompactly are those corresponding to barycentric rays. The next question is whether every barycentric horosphere admits such actions. It seems that the answer is different from case to case.

In the symmetric space $X = \mathbb{H}^2 \times \cdots \times \mathbb{H}^2$ (n times), the stabilizer of a barycentric horosphere is isomorphic to the solvable group in Example (1) above with ϕ having as image a conjugate of the group of diagonal matrices with determinant one. This group does have lattices. It may be shown by means of two arguments.

a) Let K be an algebraic totally real extension of degree n and $\mathfrak{O}_K$ its ring of integers. The group $\Gamma_K = PSL(2, \mathfrak{O}_K)$ embeds into $PSL(2,\mathbb{R})^n$ in the same way as K into $\mathbb{R}^n$, and it is a lattice in it. The group Γ_K as well as its image in $PSL(2,\mathbb{R})^n$ is called *a Hilbert modular group.* It acts on $(\mathbb{H}^2)^n$ by isometries with quotient of finite volume. It has $\mathbb{Q}$-rank one (see [Wi, Chapter 9] for a definition and a survey of the geometrical properties of $\mathbb{Q}$-rank one lattices).

In $(\mathbb{H}^2)^n$ there are countably many barycentric horospheres such that their stabilizers in Γ_K act cocompactly on them [vG, Chapter 1].

b) Consider in $SL(n+1,\mathbb{Z})$ the subgroup

$$\begin{pmatrix} SL(n,\mathbb{Z}) & \mathbb{Z}^n \\ 0 & 1 \end{pmatrix}.$$

The group $SL(n,\mathbb{Z})$ contains a free abelian subgroup Δ of rank $n-1$ which is a lattice in a conjugate D of the diagonal subgroup of $SL(n,\mathbb{R})$ [Mo, Chapter 8]. The group

$$\begin{pmatrix} D & \mathbb{R}^n \\ 0 & 1 \end{pmatrix}.$$

is isomorphic to the solvable Lie group we are studying, and by replacing D with Δ and $\mathbb{R}^n$ with $\mathbb{Z}^n$ one obtains a cocompact lattice in it.

In the symmetric space $X = SL(n,\mathbb{R})/SO(n)$, the stabilizer of a barycentric horosphere is isomorphic to a solvable group T_n as in Example (2), with $(u_1, u_2, \dots, u_n) = (n-1,\, n-2, \dots, 1,\, 0)$. By [Ti] there exist $\mathbb{Q}$-rank one lattices in $SL(3,\mathbb{R})$, so there exist lattices in T_3. From $n = 4$ on, T_n no longer has lattices. We explain in detail the case $n = 4$. If Γ is a lattice in T_4 and N is the subgroup of upper triangular matrices with 1 on the diagonal, $\Gamma \cap N$ is a lattice in N [Ra, Corollary 3.5] and $\Gamma \cap [N,N]$ is a lattice in $[N,N]$ [Ra, Corollary 1 of Theorem 2.3]. On the other hand, since Γ is cocompact, there exists $\gamma \in \Gamma$ whose semisimple part $a_\gamma = \text{diag}\,(e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}, e^{\lambda_4})$, $\sum \lambda_i = 0$, is such that $\lambda_1 + \lambda_2 \neq 0$. We have that $a(\gamma)(\Gamma \cap [N,N]) \subset \Gamma \cap [N,N]$, where $a(\gamma)$ is the conjugacy by γ. The same is true for γ^{-1}, so $a(\gamma)(\Gamma \cap [N,N]) = \Gamma \cap [N,N]$. This implies that $a(\gamma)$ preserves the Haar measure on $[N,N]$ and the same is true for $a(a_\gamma)$. On the other hand, $\det\, Ad(a_\gamma)|_{\mathfrak{n}_1} = e^{4\lambda_1 + 3\lambda_2 + \lambda_3}$, where $\mathfrak{n}_1$ is the Lie algebra of $[N,N]$, so $a(a_\gamma)$ preserves the Haar measure on $[N,N]$ if and only if $4\lambda_1 + 3\lambda_2 + \lambda_3 = 0$. This and the barycentric condition $3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ imply $\lambda_1 + \lambda_2 = 0$, which contradicts our choice of γ.

If we no longer ask that the discrete group Γ be solvable, there is a larger class of horospheres admitting a proper discontinuous cocompact action of such a group, than just the barycentric horospheres. Most of the examples come from actions of $\mathbb{Q}$-rank one lattices on symmetric spaces. As in the particular case explained in Example (a), every $\mathbb{Q}$-rank one lattice contains finitely many classes of conjugate subgroups acting properly discontinuously cocompactly on horospheres. Quite often (for example in $SL(n,\mathbb{R})$, $n \geq 4$) these horospheres are not barycentric. See [Ti] and [Wi] for a detailed discussion of the subject.

Theorem 6.9 has been obtained independently by E. Leuzinger and Chr. Pittet [LP] in the case $X = SL(n,\mathbb{R})/SO(n)$. Also, in a previous paper [Dr$_2$] we have proved that for some of the solvable groups to which Theorem 6.9 applies the filling order is at most "asymptotically cubic".

The outline of the proof of Theorem 6.9 is as follows. It is enough to prove a quadratic filling for closed curves composed of m very long geodesic segments and m comparatively short arcs joining endpoints of consecutive segments, where $m \in \mathbb{N}$ is fixed. Such a result is sufficient for a quadratic filling order since for a generic loop one can apply an induction argument [Dr$_4$, §4.D], which the author has learned from M. Sapir. With an argument similar to the one in the proof of Theorem 4.6, it is shown that it is sufficient to prove a quadratic filling for closed polygonal lines with m edges in a generic asymptotic cone. The asymptotic cone of a horosphere in X is a horosphere in the asymptotic cone of X [Dr$_1$]. On the other hand the asymptotic cone of a symmetric space X is an Euclidean building of the same rank [KlL]. An Euclidean building of rank r is a bunch of "flats", that is, of isometric copies of the Euclidean space $\mathbb{E}^r$. The trace of a horosphere in each of these flats is a polytopic hypersurface. These polytopic hypersurfaces can take, up to similarity, only a finite number of shapes. With an induction argument on m we may reduce the problem of filling a closed polygonal line with m edges in the horosphere to that of filling a loop in an Euclidean polytopic hypersurface. But every such loop has a quadratic filling.

Concerning the non-polycyclic case, G. Arzhantseva and D. Osin [AO] constructed a sequence of discrete non-polycyclic solvable groups with filling orders that are at most cubic.

References

[All] D. Allcock, *"An isoperimetric inequality for the Heisenberg groups"*, Geom. Funct. Analysis **8** (1998), 219-233.

[Al] J. Alonso, *"Inégalités isoperimétriques et quasi-isométries"*, C. R. Acad. Sci. Paris, Serie 1, **311** (1991), 761-764.

[AO] G. Arzhantseva, D. Osin, *"Isoperimetric inequalities in HNN-extensions"*, to appear in Trans. AMS.

[Bas] H. Bass, *"The degree of polynomial growth of finitely generated nilpotent groups"*, Proc. London Math. Soc. **25** (1972), 603-614.

[Bat] M. Batty, *"Groups with a sublinear isoperimetric inequality"*, IMS Bulletin **42** (1999), 5-10.

[BB] N. Brady, M. Bridson, *"There is only one gap in the isoperimetric spectrum"*, Geom. Funct. Anal. **10** (2000), 1053-1070.

[BMS] G. Baumslag, C.F. Miller, H. Short, *"Isoperimetric inequalities and the homology of groups"*. Invent. Math. **113** (1993), no. 3, 531-560.

[Bou] N. Bourbaki, "Topologie générale", quatrième édition, Hermann, Paris, 1965.

[Bow_1] B. Bowditch, *"A short proof that a sub-quadratic isoperimetric inequality implies a linear one"*, Mich. J. Math **42** (1995), 103-107.

[Bow_2] B. Bowditch, *"Notes on Gromov's hyperbolicity criterion for path metric spaces"*, "Group Theory from a Geometrical Viewpoint", E. Ghys, A. Haefliger, A. Verjovski (eds), ICTP Trieste, World Scientific Publishing Co. 1991.

[BG] M. R. Bridson, S. M. Gersten, *"The optimal isoperimetric inequality for torus bundles over the circle"*, Quart. J. Math. Oxford Ser. **47** (1996), no. 185, 1-23.

[BH] M. R. Bridson, A. Haefliger, "Metric Spaces of Non-positive Curvature", Springer, 1999.

[Br_1] M. Bridson, *"Asymptotic cones and polynomial isoperimetric inequalities"*, Topology **38** (1999), 543-554.

[Br_2] M. Bridson, *"Fractional isoperimetric inequalities and subgroup distorsion"*, Journal Amer. Math.Soc. **12** (1999), 1103-1118.

[BP] M. R. Bridson, C. Pittet, *"Isoperimetric inequalities for the fundamental groups of torus bundles over the circle"*, Geom. Dedicata **49** (1994), 203-219.

[BS_1] R. Bieri, R. Strebel, *"Almost finitely presented soluble groups"*, Comment. Math. Helv. **53** (1978), no. 2, 258-278.

[BS_2] R. Bieri, R. Strebel, *"Valuations and finitely presented metabelian groups"*, Proc. London Math. Soc. **41** (1980), no. 3, 439-464.

[BT] J. Burillo, J. Taback, *"Equivalence of geometric and combinatorial Dehn functions"*, preprint.

[Bu] J. Burillo, *"Dimension and fundamental groups of asymptotic cones"*, PhD thesis, University of Utah, june 1996.

[BW] J. Block, S. Weinberger, *"Large scale homology theories and geometry"*, in "Geometric topology"(Athens, Georgia, 1993), 522-569, AMS/IP Stud. Adv. Math., **2** (1997), Part 1, Amer. Math. Soc., Providence, RI.

[Can] J. W. Cannon, *" The combinatorial structure of cocompact discrete hyperbolic groups"*, Geom. Dedicata **16** (1984), no. 2, 123-148.

[CDP] M. Coornaert, T. Delzant, A. Papadopoulos, "Géométrie et théorie des groupes", Lecture Notes in Math. 1441, Springer 1980.

[Di] A. Dioubina, *"Instability of the virtual solvability and the property of being virtually torsion-free for quasi-isometric groups"*, preprint 1999.

[DP_1] A Dioubina, I. Polterovich, *"Explicit constructions of universal R-trees and asymptotic geometry of hyperbolic spaces"*, preprint, math.DG/9904133.

[DP_2] A Dioubina, I. Polterovich, *"Structures at infinity of hyperbolic spaces"*, (Russian) Uspekhi Mat. Nauk **53** (1998), no. 5(323), 239-240; translation in Russian Math. Surveys **53** (1998), no. 5, 1093-1094.

[Dr_1] C. Druţu, *"Nondistorsion des horospheres dans des immeubles euclidiens et dans des espaces symetriques"*, Geom. Funct. Analysis **7** (1997), no.5, 712-754.

[Dr_2] C. Druţu, *"Remplissage dans des reseaux de Q-rang* 1 *et dans des groupes resolubles"*, Pacific J. Math **185** (1998), 269-305.

[Dr_3] C. Druţu, *"Cônes asymptotiques et invariants de quasi-isometrie pour des espaces metriques hyperboliques"*, Ann. Inst. Fourier **51** (2001), no. 1, 81-97.

[Dr_4] C. Druţu, *"Filling in solvable groups and in lattices in semisimple groups"*, preprint 2000.

[Dr_5] C. Druţu, *"Quasi-isometric classification of non-uniform lattices in semisimple groups of higher rank"*, Geom. Funct. Anal. **10** (2000), no. 2, 327-388.

[ECHPT] D. B. A. Epstein, J. W. Cannon, D. F. Holt, M. S. Paterson, W. P. Thurston, "Word processing and group theory", Jones and Bartlett Publ., 1992.

[FM_1] B. Farb and L. Mosher, *"A rigidity theorem for the solvable Baumslag-Solitar groups. With an appendix by Daryl Cooper"*, Invent. Math. **131** (1998), no. 2, 419-451.

[FM_2] B. Farb and L. Mosher, *"Quasi-isometric rigidity for the solvable Baumslag-Solitar groups, II"*, Invent. Math. **137** (1999), no. 3, 613-649.

[FM_3] B. Farb and L. Mosher, *"On the asymptotic geometry of abelian-by-cyclic groups"*, Acta Math. **184** (2000), no. 2, 145-202.

[FM_4] B. Farb, L. Mosher, *"Problems on the geometry of finitely generated solvable groups"*, in "Crystallographic Groups and their Generalizations (Kortrijk, 1999)", (ed. P. Igodt, et al.), Cont. Math. 262, Amer. Math. Soc. (2000).

[Ge_1] S. M. Gersten, *"Isoperimetric and Isodiametric Functions of Finite Presentations"*, "Geometric Group Theory"(vol. 1), G. A. Niblo, M. A. Roller (eds), Proceedings of the Symposium held in Sussex, LMS Lecture Notes Series 181, Cambridge University Press 1991.

[Ge_2] S. M. Gersten, *"Quasi-isometry invariance of cohomological dimension"*, C. R. Acad. Sci. Paris **316** (1993), Série I, 411-416.

[Ge_3] S. M. Gersten, *"Dehn functions and l_1-norms of finite presentations"*, in "Algorithms and classification in combinatorial group theory", (Berkeley, 1989), 195-224, Math. Sci. Res. Inst. Publ., 23, Springer, New York, 1992.

[GHR] S. M. Gersten, D. F. Holt, T. R. Riley, *"Isoperimetric inequalities for nilpotent groups"*, preprint, 2002.

[GH_1] E. Ghys, P. de la Harpe (eds), "Sur les groupes hyperboliques d'apres M. Gromov", Birkhäuser, 1990.

[GH_2] E. Ghys, P. de la Harpe, *"Infinite groups as geometric objects (after Gromov)"*, "Ergodic theory, symbolic dynamics, and hyperbolic spaces", (Trieste, 1989), (ed. T. Bedford, M. Keane, C. Series), 299-314, Oxford Sci. Publ., Oxford Univ. Press, New York, 1991.

[GLP] M. Gromov, J. Lafontaine, P. Pansu, "Structures metriques pour les varietes riemanniennes", Cedic/Fernand Nathan, Paris (1981).

[Gr_1] M. Gromov, *"Infinite groups as geometric objects"*, Proceedings of the International Congress of Mathematicians, (Warsaw, 1983), 385-392.

[Gr_2] M. Gromov, *"Groups of polynomial growth and expanding maps"*, Publ. Math. IHES **53** (1981), 53-73.

[Gr_3] M. Gromov, *"Hyperbolic groups"*, "Essays in group theory", S.Gersten (ed), MSRI Publ. **8**, Springer (1987), 75-265.

[Gr_4] M. Gromov, *"Asymptotic Invariants of Infinite Groups"*, Geometric Group Theory(vol. 2), G. A. Niblo, M. A. Roller (eds), Proc. of the Symposium held in Sussex, LMS Lecture Notes Series 181, Cambridge University Press 1991.

[KaL_1] M. Kapovich, B. Leeb, *"On asymptotic cones and quasi-isometry classes of fundamental groups of nonpositively curved manifolds"*, Geom. Funct. Analysis **3** (1995), vol.5, 582-603.

[KaL_2] M. Kapovich, B. Leeb, *" Quasi-isometries preserve the geometric decomposition of Haken manifolds"*, Invent. Math. **128** (1997), no.2, 393-416.

[Kh_1] O. G. Kharlampovič, *" A finitely presented solvable group with unsolvable word problem"*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), no. 4, 852-873; translation in Math. USSR Izvestija **19** (1982), no.1, 151-169.

[KhS] O. G. Kharlampovič, M. V. Sapir *"Algorithmic problems in varieties"*, Int. J. Algebra Comput. **5** (1995), no.4-5, 379-602.

[KlL] B. Kleiner and B. Leeb, *"Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings"*, Publ. Math. IHES **86** (1997), 115-197.

[LP] E. Leuzinger, Ch. Pittet, *"Quadratic Dehn functions for solvable groups"*, preprint 2000.

[Mi] J. Milnor, *"Growth of finitely generated solvable groups"*, J. Differential Geometry **2** (1968), 447-449.

[MNO] J. C. Mayer, J. Nikiel, L. G. Oversteegen, *"Universal spaces for $\mathbb{R}$-trees"*, Trans. Amer. Math. Soc. **334** (1992), no. 1, 411-432.

[Mo] G. D. Mostow, Strong Rigidity of Locally Symmetric Spaces, AMS Studies no.78, Princeton University Press, 1973.

[Ol] A. Yu. Olshanski, *"Hyperbolicity of groups with subquadratic isoperimetric inequalities"*, Intl. J. Alg. Comp. **1** (1991), 282-290.

[OS] A. Yu. Olshanskii, M. V. Sapir, *"Quadratic isoperimetric functions of the Heisenberg groups. A combinatorial proof"*, Algebra, 11, J. Math. Sci. (New York) **93** (1999), no. 6, 921-927.

[P_1] P. Papasoglu, *"An algorithm detecting hyperbolicity"*, in "Geometric and computational perspectives on infinite groups" (Minneapolis, MN and New Brunswick, NJ, 1994), 193-200, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **25**, Amer. Math. Soc., Providence, RI, 1996.

[P_2] P. Papasoglu, *"On the subquadratic isoperimetric inequality"*, Geometric group theory, R. Charney, M. Davis, M. Shapiro (eds), de Gruyter, Berlin-New-York, 1995.

[P_3] P. Papasoglu, private communications.

[P_4] P. Papasoglu, *"On the asymptotic cone of groups satisfying a quadratic isoperimetric inequality"*, J. Diff. Geometry **44** (1996), 789-806.

[P_5] P. Papasoglu, *"Quasi-flats in semihyperbolic groups"*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1267-1273.

[P_6] P. Papasoglu, *"Isodiametric and isoperimetric inequalities for complexes and groups"*, J. London Math. Soc. **62** (2000), no. 2, 97-106.

[Pa_1] P. Pansu, *"Croissance des boules et des geodesiques fermees dans les nilvarietes"*, Ergod. Th. Dynam. Syst. **3** (1983), 415-445.

[Pa_2] P. Pansu, *"Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un"*, Ann. of Math. **129** (1989), no. 1, 1-60.

[Pau] F. Paulin, *"Topologie de Gromov equivariante, structures hyperboliques et arbres reels"*, Invent. Math. **94** (1988), 53-84.

[Pi_1] Ch. Pittet, *"Isoperimetric inequalities for homogeneous nilpotent groups"*, in: Geometric group theory (Columbus, OH, 1992), Ohio State Univ. Math. Res. Inst. Publ., 3, de Gruyter, Berlin, 1995, 159-164.

[Pi_2] Ch. Pittet, *"Isoperimetric Inequalities in Nilpotent Groups"*, J. London Math. Soc. **55** (1997), 588-600.

[PS_1] I. Polterovich, A. Shnirelman, *"On the asymptotic geometry of the hyperbolic plane"*, preprint, math.DG/9806106.

[PS_2] I. Polterovich, A. Shnirelman, *"An asymptotic subcone of the Lobachevskii plane as a space of functions"*, Uspekhi Mat. Nauk **52** (1997), no. 4(316), 209-10; translation in Russian Math. Surveys **52** (1997), no. 4, 842-843.

[Ra] M. S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, 1972.

[SBR] M. Sapir, J. C. Birget, E. Rips, *"Isoperimetric and isodiametric functions of groups"*, to appear in Ann. of Math.

[Šv] A. S. Švarč, *"A volume invariant of coverings"*, (Russian), Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 32-34.

[ThV] S. Thomas, B. Velickovic, *"Asymptotic cones of finitely generated groups"*, Bull. London Math. Soc **32** (2000), 203-208.

[Ti] J. Tits, *"Classification of algebraic semisimple groups"*, in: Algebraic groups and discontinuous subgroups, Proc. Symp. Pure Math., vol 9, 33-62, Boulder 1965, A. M. S. 1966.

[VDW] L. van den Dries, A. J. Wilkie, *"On Gromov's theorem concerning groups of polynomial growth and elementary logic"*. J. of Algebra **89** (1984), 349-374.

[vG] G. van der Geer, Hilbert Modular Surfaces, Springer, 1988.

[Wi] D. Witte, "Introduction to Arithmetic groups", http://www.math.okstate.edu/ ∼dwitte.

[Wo] J. A. Wolf, *"Growth of finitely generated solvable groups and curvature of Riemanniann manifolds*, J. Differential Geometry **2** (1968), 421-446.