Th. G nilpotent, class k

$C^k G$ last non-trivial subgp. in lower central series.

$\forall\, h \in C^k G$,

$$|h|_{C^k G} \asymp |h|_G^k$$

Cay(G)

Proof $\boxed{\geq}$

$\boxed{\leq}$ **Remark** Let X finite generating G.

We can extend it to $\hat{X}$ finite gen., moreover s.t. $\forall x, y \in \hat{X}$, $[x,y] \in \hat{X}$.

Note $\hat{X}$ contains all the i-fold left commutators in X, $\forall i$:

$$[\,..[[x_1,x_2],x_3],\dots,x_i]$$

lcs-generating set = a set S s.t. $S_i = S \cap C^i G$ generates $C^i G$. Let $S_i' = S \setminus S_{i+1}$.

w word in S:

- length $|w|$
- i-length $|w|_i$ = nb. of letters in S_i' that occur in w.
- lcs-length = $(|w|_1, |w|_2, \dots, |w|_k)$.

$g \in G$ has lcs-length at most $(n_1, .., n_k)$ when g represented by w with $|w|_i \leq n_i$.

$|g|_S^{lcs} \preccurlyeq (n_1, \dots, n_k)$ = $\exists\, \lambda$ constant s.t. lcs-length is at most $(\lambda n_1, \dots, \lambda n_k)$.

Prove by induction on i: $\exists$ an lcs-generating set $S^{(i)}$ and a constant λ_i s.t. $\forall\, g \in C^k G$

$$|g|^{lcs}_{S^{(i)}} \leq (\underbrace{0, \dots, 0}_{i}, \lambda_i |g|_X^{i+1}, \dots, \lambda_i |g|_X^k)$$

$i=0 \quad X \to \hat{X}$

$S^{(0)} = \hat{X}$. Denote $S^{(0)}$ by S.

$$G/C^2G \simeq \mathbb{Z}^m \times \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_q}$$

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$$

Take $t_1, \dots, t_m$ projecting on gen. of $\mathbb{Z}^m$

Take $\tau_1, \dots, \tau_q$ proj. onto gen of $\mathbb{Z}_{n_i}$.

$\tau_i C^2G$ generates $\mathbb{Z}_{n_i} \Rightarrow \tau_i^{n_i} \in C^2G$

$Z = S_2 \cup \{t_1, \dots, t_m, \tau_1, \dots, \tau_q\}$, $Y = \hat{Z}$

<u>Hypothesis</u> g written as w in S s.t. the lcs-length $\leqslant (n, n^2, \dots, n^k)$,

Here $n = |g|_X$

We replaced S_1' by $\{t_1, \dots, t_m, \tau_1, \dots, \tau_q\}$ + added commutators.

1-length $\leqslant \lambda \cdot$ old 1-length

i-length $\leqslant$ old i-length $+ \lambda \cdot$ old 1-length

We still have, for the new word w' representing g that its lcs-length is $\preccurlyeq (n, n^2, \dots, n^k)$.

Now we move all $t_i^{\pm 1}$, $\tau_j^{\pm 1}$ in front of the word.

Assume $t_1^{\pm 1}$ occurs ℓ times, $\ell \leqslant n$.

By induction on $j \leq \ell$ we construct word w_j s.t. $w_0 = w'$, and:

(1) $w_j = t_1^{\alpha_j} u_j$ with $|\alpha_j| \leq j$.

(2) in u_j, $t_1^{\pm 1}$ occur $\ell - j$ times.

(3) if w_j has lcs-length $\leq (n_1, n_2, \dots, n_k)$ then

lcs-length of w_{j+1} is $\preccurlyeq (n_1, n_2 + n_1, \dots, n_k + n_{k-1})$

$j \to j+1$:

find first occurrence of $t_1^{\pm 1}$ in u_j, move it to the left by:

$$x t_1 = t_1 x [x^{-1}, t_1^{-1}]$$

In the end: $w_\ell = t_1^{\alpha} u_\ell$,

$$|w_\ell|_i \leq |w'|_i + \ell |w'|_{i-1} + \ell^2 |w'|_{i-2} + \dots + \ell^{i-1} |w'|_1 \preccurlyeq n^i.$$

We continue, move $t_2, \dots, t_m, z_1, \dots, z_q$ to the left, in that order.

We obtain w'' of lcs-length $\preccurlyeq (n, n^2, \dots, n^k)$ and

$w'' = t_1^{\alpha_1} \dots t_m^{\alpha_m} z_1^{\beta_1} \dots z_q^{\beta_q} u$ representing $g \in C^k G$.

Project on $G / C^2 G \simeq \mathbb{Z}^m \times \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_q}$

g becomes $0 \Rightarrow \alpha_i = 0$, β_j multiple n_j

$$\beta_j = n_j \delta_j$$

$$w'' = (z_1^{n_1})^{\delta_1} \dots (z_q^{n_q})^{\delta_q} u$$

Add to Y the generators $\theta_j = z_j^{n_j} \in C^2 G$.

$\to Y_1$. Take $S^{(1)} = \hat{Y}_1$.

For $i \Rightarrow i+1$, similar argument.

For $i = k$, $\exists$ lcs-gen. $S^{(k)}$ and λ_k s.t.

$\forall g \in C^k G$,

$$|g|^{lcs}_{S^{(k)}} \preccurlyeq (0, \dots, 0, \lambda_k n^k)$$

$T = S^{(k)} \cap C^k G$ generates $C^k G$

$$|g|_T \leq \lambda_k \cdot n^k. \qquad \boxed{\exists}$$

__Corollary__ G nilpotent f.g.,

$H \leq G$, $\forall h \in H$

$$|h|_G \leq |h|_H \leq |h|_G^m, \ m \in \mathbb{N}.$$

__Proof__ Induction on nilp. class.

C^iG, $i < k$, $C^iG \geq C^kG$.

The minimal m is rational, obtained:

$\forall h \in H$, $\partial_G(h) =$ maximal i s.t. C^iG contains h

$\partial_H(h) =$ — " — j s.t. $h \in C^jH$

$$\sup_{h \in H} \frac{\partial_G(h)}{\partial_H(h)} = m.$$

Bass Theorem

Growth function of a gp. G + word m.:

$$\mathcal{G}_G(n) = \# B(1, n).$$

Let G be nilpotent of class k,

$$C^iG / C_k^{i+1}G \simeq \mathbb{Z}^{m_i} \times \text{finite}.$$

$$d = \sum_{i=1} i\, m_i$$

Then $\mathcal{G}_G(n) \asymp n^d$.

__Proof__ by induction on class k.

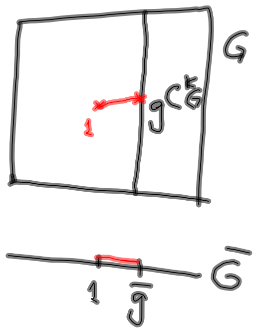
$k = 1 \Rightarrow G$ abelian

Assume true for $k-1$, consider G k-step nilpotent.

Take $\overline{G} = G / C^kG$ $(k-1)$-step nilp.

__Case 1__ C^kG finite $\Rightarrow G$ quasi-isom. to $\overline{G}$, apply the inductive hyp.

Case 2 $C^kG \simeq \mathbb{Z}^{m_k} \times \text{finite},\ m_k \geq 1.$



$|\bar{g}|_{\bar{G}} \leq R \iff$

$\iff$ in G the coset gC^kG intersects $B(1,R)$.

Lower bound $d_1 = d - k m_k$.

$\mathcal{G}_{\bar{G}}(n) \asymp n^{d_1}$, by hyp.

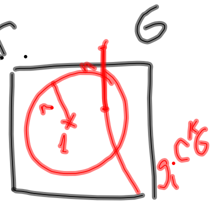
In particular there exist at least $N_1 = \lambda \cdot n^{d_1}$ elements in G, of length $\leq n$, not congruent modulo C^kG.

$C^kG \cap B_G(1,n)$ is approximately same as $C^kG \cap B_{C^kG}(1,n^k)$.

$C^kG \simeq \mathbb{Z}^{m_k} \times \text{finite}$

$\tilde{B} = C^kG \cap B_{C^kG}(1,n^k)$ contains n^{km_k}.

We had $g_1, g_2, \dots, g_{N_1}$ in $B(1,n)$, non-congruent mod C^kG.



$\forall i$, $g_i \tilde{B}$ is $B_G(1,2n)$

We obtain in $B(1,2n)$ $N_1 \cdot n^{km_k}$ distinct elements.

$\mathcal{G}_G(2n) \geq \lambda \cdot n^{d_1} \cdot n^{km_k} = \lambda n^d$.

Upper bound $\exists$ at most $N = \mu n^{d_1}$ cosets gC^kG intersecting $B_G(1,n)$.

$\forall g \in B_G(1,n)$, $g \in g_i C^k G$,

where $g_1, g_2, \dots, g_N$ is a maximal set in $B(1,n)$ of elem. $C^k G$-non-congruent.

$\Rightarrow g_i^{-1} g \in C^k G$, $|g_i^{-1} g| \leq |g_i^{-1}| + |g|$
$\lesssim 2n$

$\Rightarrow g_i^{-1} g \in C^k G \cap B_G(1, 2n) \asymp$
$\asymp C^k G \cap B_{C^k G}(1, (2n)^k)$

Thus in $B_G(1,n)$ we have at most:

$$N \cdot (2n)^{k m_k} \leq \mu\, n^{d_2} (2n)^{k m_k} \lesssim n^d. \quad \blacksquare$$

Nilpotent $\subseteq$ Polycyclic $\subseteq$ Solvable

<u>Wolf Theorem</u>

Every polycyclic group is
(1) either virtually nilpotent
(2) or of exponential growth.

<u>Proof</u> $\forall$ polycyclic group is virtually poly-C_∞.

Polycyclic = G having a descending subnormal finite series:

$$G = N_0 \triangleright N_1 \triangleright N_2 \triangleright \dots \triangleright N_k \triangleright N_{k+1} = \{1\}$$

$\forall i$, $N_{i+1} \triangleleft N_i$ and N_i / N_{i+1} cyclic.

if $\forall i$, $N_i / N_{i+1} \simeq \mathbb{Z} \Rightarrow$ poly-C_∞.

$1 \to N_{i+1} \to N_i \to \mathbb{Z} \to 1 \Rightarrow N_i = N_{i+1} \rtimes \mathbb{Z}$.

$N_k = \mathbb{Z}$ $\quad N_{k-1} = \mathbb{Z} \rtimes \mathbb{Z}$.

Proof by ind. on length K of series.

$K = 1 \Rightarrow G$ cyclic.

Assume true for K, consider G:

$$G = N_0 \rhd N_1 \rhd \dots \rhd N_{K+1} \rhd N_{K+2} = \{1\}.$$

Take $N_1 \Rightarrow N_1$ virt. nilpotent or expo. growth.

N_1 expon. growth $\mathcal{G}_{N_1}(n) \simeq a^n$
$\Rightarrow \mathcal{G}_G(n) \geq a^n$.

$\forall h \in N_1$, $|h|_G \leq |h|_{N_1} \leq n$

Remark Every f.g. has growth $\leq a^n$.

Assume N_1 virtually nilpotent:
$\exists K \leq N_1$ subgp. of finite index,
K nilpotent.

$$G = N_1 \rtimes \mathbb{Z}$$

Two standard results H f.g. group.
(1) $\forall n$, H has finitely many subgroups of index n.
(2) $\forall K \leq H$, K f.i., $\exists C$ characteristic
$C \leq K$, C f.i. G.

Take $C \leq K \leq N_1$, C characteristic

Take $C \rtimes \mathbb{Z}$, has finite index
in $G = N_1 \rtimes \mathbb{Z}$.

To finish proof we need:
If $G = C \rtimes \mathbb{Z}$, C nilpotent then:
(1) either G virt nilp.
(2) expo growth.