Thm A polycyclic gp is either virtually nilpotent or of expon. growth.

Proof reduced to

Prop. Consider K nilpotent f.g.
A group $G = K \rtimes \mathbb{Z}$ is:
(1) either virtually nilp;
(2) or of expon. growth.

Remark If G is f.g. its growth is at most exponential, i.e. $\mathcal{G}_G(n) \preccurlyeq e^n$:
Because:
(1) $\forall$ f.g. group is quotient of a f.g. free group.
(2) the growth of a f.g. free gp. is $\preccurlyeq e^n$.

In Prop., $G = K \rtimes \mathbb{Z}$

$$\mathbb{Z} \longrightarrow Aut(K)$$
$$1 \longmapsto \varphi \quad ; \quad n \longmapsto \varphi^n$$

Notation: $K \rtimes_\varphi \mathbb{Z}$

Properties of autom. of abelian gps.

(1) The gp. of autom. of $\mathbb{Z}^n$ is
$$GL(n,\mathbb{Z}) = \{ M \in M(n,\mathbb{Z}) ; \det M = \pm 1 \}.$$

(2) If M has all eigenvalues in $\{ z \in \mathbb{C} ; |z| = 1 \}$ then the eigenvalues are roots of unity: $\lambda^k = 1$.

(3) If M has only eigenvalue 1 then $\exists P \in GL(n,\mathbb{Z})$ s.t.
$$M = P \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} P^{-1}.$$

Equivalently there exists a descending series
$\Lambda_0 = \{0\} \leq \Lambda_1 \leq \dots \leq \Lambda_i \leq \Lambda_{i+1} \leq \dots \leq \Lambda_n$ s.t.

- $\Lambda_i \simeq \mathbb{Z}^i$
- $\Lambda_{i+1} / \Lambda_i \simeq \mathbb{Z}$
- $M(\Lambda_i) = \Lambda_i$
- M induces on Λ_{i+1}/Λ_i the id.

(4) Assume M has one eigenvalue λ s.t. $|\lambda| \geq 2$. Then $v \in \mathbb{Z}^n$ s.t. the following map is inj.:

$$\bigoplus_{\substack{m \in \mathbb{Z} \\ m \geq 0}} (\mathbb{Z}_2)_m \longrightarrow \mathbb{Z}^n$$

$$(s_m) \longmapsto s_0 \cdot v + s_1 M v + \ldots + s_i M^i v + \ldots$$

Proof $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$, $\varphi(u) = Mu$.

Extends to $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, has an adjoint transf. $\varphi^* : (\mathbb{C}^n)' \to (\mathbb{C}^n)'$,
$f \mapsto f \circ \varphi$

φ^* has matrix $M^T \Rightarrow$ has eigenval. λ:
$\exists f : \mathbb{C}^n \to \mathbb{C}$, $f \not\equiv 0$ s.t.

$$f \circ \varphi = \lambda f$$

Pick $v \in \mathbb{Z}^n \setminus \mathrm{Ker} f$.

If the map corresponding to v not inj then $\exists$ $t_i \in \{0, \pm 1\}$, $N \geq 1$ s.t.

$$t_0 v + t_1 M v + \ldots + t_N M^N v = 0 \quad (*)$$

Assume $t_N \neq 0$. We re-write $(*)$:

$$M^N v = r_0 v + r_1 M v + \ldots + r_{N-1} M^{N-1} v,$$
$$r_i \in \{0, \pm 1\}.$$

Apply f: $(f(Mv) = \lambda f(v))$.

$$\lambda^N f(v) = (r_0 + r_1 \lambda + \ldots + r_{N-1} \lambda^{N-1}) f(v)$$

$$\Rightarrow |\lambda|^N \leq 1 + |\lambda| + \ldots + |\lambda|^{N-1} =$$
$$= \frac{|\lambda|^N - 1}{|\lambda| - 1} \leq \lambda^N - 1. \quad \#$$

Proof of Prop.

$\varphi : K \to K$, K nilpotent.

$$\varphi(C^i K) = C^i K$$

$$Q_i = C^i K / C^{i+1} K \simeq \mathbb{Z}^{m_i} \times F_i$$

$$F_i = \mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_k} \quad (\text{finite}).$$

φ induces autom. φ_i of $Q_i \Rightarrow$

$\Rightarrow$ induces autom. $\overline{\varphi}_i$ of $\mathbb{Z}^{m_i}$,

$\tilde{\varphi}_i$ of F_i.

$\overline{\varphi}_i \longrightarrow M_i \in GL(m_i, \mathbb{Z})$.

Case (1) All M_i have eigenvalues on $\{z ; |z| = 1\}$.

Goal: Then G virt. nilp.

$\exists N$ s.t. • all M_i^N have eigenvalue 1.

• $\tilde{\varphi}_i^N = id_{F_i}$.

$G = K \rtimes_\varphi \mathbb{Z}$ has the finite index subgp. $K \rtimes_\varphi (N\mathbb{Z}) \simeq K \rtimes_{\varphi^N} \mathbb{Z}$.

Thus, up to replacing G by fin. ind. subgp., we may assume all M_i have eigenvalue (1), all $\tilde{\varphi}_i = id_{F_i}$.

(3) $\Rightarrow \exists$ a descending subnormal series in K as follows:

$H_0 = K \rhd H_1 \rhd \dots \rhd H_m = \{1\}$ s.t.

- H_i / H_{i+1} cyclic;
- $\varphi(H_i) = H_i$;
- φ induces the id on each H_i/H_{i+1};
- all $\Gamma^j K$ appear among H_i.

$G = K \rtimes_\varphi \mathbb{Z}$. Denote by t a gen. of $\mathbb{Z}$.

$\forall k \in K \quad tkt^{-1} = \varphi(k)$

$tH_it^{-1} = H_i$

$\forall h \in H_i \quad t^k h t^{-k} H_{i+1} = h H_{i+1} \Leftrightarrow$

$\Leftrightarrow [t^k, h] \in H_{i+1}$

$G = K \rtimes \mathbb{Z} \Rightarrow 1 \to K \to G \to \mathbb{Z} \to 1$

$\Rightarrow C^2 G = [G, G] \leq K = H_0$

$\cdot\ C^3 G = [G, C^2 G] \leq [G, H_0]$.

We prove that $[G, H_i] \leq H_{i+1}$.

$\forall h \in H_i, \forall g \in G, g = k t^a, a \in \mathbb{Z}$.

$[k t^a, h] = k t^a h t^{-a} k^{-1} h^{-1} =$

$= [k, \underbrace{t^a h t^{-a}}_{h'}]\ t^a h t^{-a} h^{-1} =$

$= [k, h'] \cdot \underbrace{[t^a, h]}_{\in H_{i+1}}$.

$k \in K,\ h' \in H_i$

$\exists j \geq 1$ s.t. $C^j K \geqslant H_i \triangleright H_{i+1} \geqslant C^{j+1} K$

$[k, h'] \in [K, H_i] \leq [K, C^j K] = C^{j+1} K \leq$
$\leq H_{i+1}$.

Thus $\leadsto [G, H_i] \leq H_{i+1}$.

$C^3 G \leq [G, H_0] \leq H_1$.

By ind. $C^{j+2} G \leq H_j$.

In part. $C^{m+2} G \leq H_m = \{1\}$. ∎

<u>Case (2)</u> $\exists M_i$ with an eigenvalue λ s.t. $|\lambda| > 1$.

By replacing φ with φ^N, we may ass. $|\lambda| > 2$!

$\exists v \in \mathbb{Z}^{m_i}$ s.t. (4) holds.

Goal: G has exponential growth.

Pick $\bar{g} \in C^i G / C^{i+1} G \simeq \mathbb{Z}^{m_i} \times F_i$ s.t.

$\bar{g} = (v, 0)$.

Take $(s_m) \in \bigoplus_{\substack{m \in \mathbb{Z} \\ m \geq 0}} (\mathbb{Z}_2)_m$

We are looking for an element in G with image in $\mathbb{Z}^{m_i}$:

$$s_0 v + s_1 M v + \ldots + s_j M^j v + \ldots$$

$$g^{s_0} \, t g^{s_1} t^{-1} \, t^2 g^{s_2} t^{-2} \ldots\ldots$$

is such an element.

Take (s_n) with $s_n = 0$ for $n \geqslant N+1$.

We have words:

$$g^{s_0} t g^{s_1} t g^{s_2} \ldots\ldots t g^{s_N} t^{-N},$$

one for every choice of (s_i).

All words have length $\leq 2N+1$ (if g, t, gt are in the gen. set).

We have 2^{N+1} distinct words.

Thus $\mathfrak{g}_G(2N+1) \geqslant 2^{N+1}$. ∎

poly-Γ_∞ = G obtained by:

$$(\ldots((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \mathbb{Z} \ldots) \rtimes \mathbb{Z}$$

Solvable groups

G. Define the iterated commutator subgroups:

$$G^{(0)} = G, \quad G^{(1)} = G' = [G, G],$$

$$G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

$$G^{(0)} \rhd G^{(1)} \rhd \ldots \rhd G^{(n)} \rhd \ldots$$

All $G^{(i)}$ are characteristic.

G is solvable if $\exists\, k$ s.t. $G^{(k)} = \{1\}$.

The minimal such k = derived length.

If derived length ≤ 2 = G is metabelian = abelian-by-abelian:

$G^{(2)} = \{1\} \lhd G'$ abelian, G/G' abelian.

Properties

(1) $H \leq G$ solvable $\Rightarrow H$ solvable.

$$H^{(n)} \leq G^{(n)}$$

(2) G solvable $\Rightarrow G/N$ solv., $\forall N \trianglelefteq G$.

(3) solvable-by-solvable $\Rightarrow$ solvable.
Not true for nilpotent:

$$\mathbb{Z}^2 \rtimes_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}.$$

(4) polycyclic $\Rightarrow$ solvable.
Induction on a minimal length of a polycyclic series + (3).

Prop. A solvable group G is polycyclic iff every subgroup of G is f.g.

Proof $\Rightarrow$ is immediate.
$\Leftarrow$: Let G be solvable + $\forall H \leq G$, H is f.g.
Induction on the derived length:
$k = 1 \Rightarrow G$ abelian, f.g. $\Rightarrow$ polyc.
Assume true for k, let G have derived length $k+1$.
G' solv. of derived length k + $\forall$ subgp. is f.g. $\Rightarrow G'$ polyc.
G/G' abelian f.g. $\Rightarrow$ polyc.
G is polyc.-by-polyc. $\Rightarrow$ polyc.

Q If G nilp., what rel. between derived length and class?

Prop. (1) $G^{(i)} \leq C^{2^i} G$, $\forall i$.
(2) If G is k-step nilp. then derived length $\leq \log_2 k$.

Proof (1) by ind. on i: $i=0$: $G = G_{1}$
$G^{(i+1)} = [G^{(i)}, G^{(i)}] \leq [C^{2^i} G, C^{2^i} G] \leq C^{2^{i+1}} G$ ∎

Rem. No lower bound in (2).

Ex. Consider the dihedral group D_{2n} = the gp. of isom. of the n-regular polygon. $D_{2n} \simeq \mathbb{Z}_n \rtimes \mathbb{Z}_2$

$n = 2^k \Rightarrow D_{2n}$ is k-nilpotent.
- derived length 2.

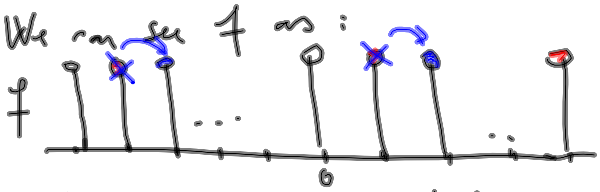
Ex. of solvable non-polyc. gp. = the lamplighter group.

$$\bigoplus_{n\in\mathbb{Z}} (\mathbb{Z}_2)_n \rtimes_\varphi \mathbb{Z} = L$$

$$\bigoplus_{n\in\mathbb{Z}} (\mathbb{Z}_2)_n = \{ f : \mathbb{Z} \to \mathbb{Z}_2 \,;\, f(m) \neq 0 \text{ for finitely many } m \}.$$

$$\mathbb{Z} \longrightarrow \mathrm{Aut}(\oplus \ldots)$$
$$1 \longmapsto \varphi$$
$$\varphi(f)(x) = f(x-1)$$

We can see f as:



L is abelian-by-abelian $\Rightarrow$ metabelian.

Ex. L is generated by:
$$(\underbrace{0}_{\text{fct.}}, 1), \quad (f_0, 0), \quad f_0(m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

L is not polycyclic.
$\bigoplus_{\mathbb{Z}} (\mathbb{Z}_2)_n$ not f.g.

Ex. $L' = \{ (f, 0) \,;\, \# \operatorname{supp} f \text{ is even} \}$.

Ex. Provide an ex. of solv. gp. of derived length 24.