Rem. $SL(n,\mathbb{K}) \subseteq GL(n,\mathbb{R})$ alg.

$$GL(n,\mathbb{R}) \hookrightarrow SL(n+1,\mathbb{R})$$

$$A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & \frac{1}{\det A} \end{pmatrix}$$

So may switch from $GL(n,\mathbb{K})$ to $SL(n,\mathbb{K})$.

~~Zariski topology~~

An algebra is noetherian if one of the following two equivalent statem. is true:

(1) every ideal is fin. generated;

(2) the set of ideals satisfies the ascending chain cond. (ACC):

$\forall\ I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq I_{n+1} \subseteq \dots$ stabilizes:

$$I_n = I_N, \ \forall n \geq N.$$

Hilbert ideal basis thm. $\Rightarrow \mathbb{K}[X_1,\dots,X_n]$ noeth. $\forall\, \mathbb{K}$ field.

$V = \mathbb{K}^n$ $\quad F$ closed $\equiv$ set of zeroes of polyn.

F closed $\longleftrightarrow$ ideal I_F of $\mathbb{K}[X_1,\dots,X_n]$

$$I_F = \{ f \ ; \ f|_F = 0 \}$$

$\mathbb{K}[X_1,\dots,X_n]$ noetherian implies:

(a) every F defined by finitely many eq.

(b) the closed sets satisfy the descending chain cond. (DCC):

$F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq F_{n+1} \supseteq \quad \Rightarrow$

$\Rightarrow F_n = F_N, \ \forall n \geq N$, for some N.

A topol. space s.t. closed sets have DCC = = noetherian

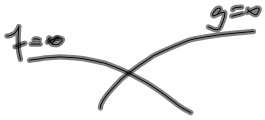
A strong version of connectedness:

A noetherian top. space is irreducible if one of the following 2 equiv. cond. holds:

(1) $\forall\, U, V$ open, non-empty $\Rightarrow U \cap V \neq \emptyset$;

(2) M cannot be = union of 2 proper closed subsets.

Ex. ① $\mathbb{K}^n$ + Zariski is irred.



(2) F Zariski closed + induced Zar.

F irred. $\Leftrightarrow \mathbb{K}[X_1,..,X_n]/I_F$ no zero divisors.

Thm. Let X noetherian top.
Then X can be written as $\bigcup_{i=1}^{n} M_i$,
where M_i irred., $M_i \not\subseteq M_j$ if $i \neq j$,
$\{M_1,..,M_n\}$ unique.

Ex. F Zariski closed + ind. top.
Thm. easy to see in this case.

Hint: F = set of zeroes of $f = gh \Rightarrow$
$\Rightarrow F = \{$zeroes of $g\} \cup \{$zeroes of $h\}$.

Thm. Let $G \leq GL(n,\mathbb{K})$ algebraic.
G is irreducible $\Leftrightarrow$ G connected in standard top.

Reference: Onishchik–Vinberg.

Cor. Every $G \leq GL(n,\mathbb{K})$ algebraic has finitely many conn. comp., all irred.

Proof of Tits Theorem

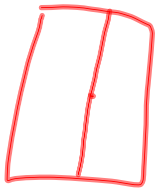
Denote $\Gamma \leq GL(n,\mathbb{K})$.
Consider $G = \overline{\Gamma}^{\text{Zariski}}$
G_0 = conn. comp. of id.
$\Gamma_0 = \Gamma \cap G_0$ finite index in Γ.

We also want that G acts on
$V = \mathbb{K}^n$ irreducibly: no proper subsp. is G-invariant.

Assume action not irred.: $\exists W \subsetneq V$ s.t. $G(W) = W$.
G acts on W, on V/W. Induct. arg.?

After repeating this a maximal number of times we obtain a flag of subspaces of $\mathbb{K}^n$: $V_0 = \{0\} \leq V_1 \leq V_2 \leq \ldots \leq V_m = \mathbb{K}^n$

s.t. $G(V_i) = V_i$, G acts irred. on all V_{i+1}/V_i, $i \geq 0$.



If $\dim V_{i+1}/V_i = 1$, $\forall i$:

Pick $\{e_1\}$ basis for V_1

$\{e_1, e_2\}$ — " — V_2

$\vdots$

$\{e_1, \ldots, e_n\}$ — " — V_n

In this basis:

$$G \leq \left\{ \begin{pmatrix} * & * & & \\ 0 & * & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & * \end{pmatrix} \leq GL(n, \mathbb{K}) \right\} = T_n.$$

Ex. $(T_n)' = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ $\quad$ $(T_n)^{(k)} = \left\{ \begin{array}{c} \scriptstyle 1\,2\,\ldots\,k \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ & \ddots & & & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} \end{array} \right\}$

$(T_n)^{(n)} = \{Id\}$

Then G and Γ solvable.

Assume that some $V_{i+1}/V_i = W_i$ have dim. ≥ 2.

We can define $\rho_i : G \to GL(W_i)$

We look at $\rho_i(\Gamma)$.

It suffices to prove thm. for $\rho_i(\Gamma)$:

* If $\forall i$, $\rho_i(\Gamma)$ are solvable then Γ is solvable.

* If $\exists i$ s.t. $\rho_i(\Gamma)$ contains F_2, then Γ contains F_2 ($\rho_i(\Gamma)$ is quot. of Γ).

Thus assume that G acts irreducibly on $V \simeq \mathbb{K}^n$.

Hypotheses: Γ, $G = \overline{\Gamma}$ in $GL(n, \mathbb{K}) = GL(V)$, G connected, G acts on $\mathbb{K}^n$ irreducibly.

Conventions: $\forall A \subseteq \mathbb{K}^n$, we denote

$$P = P(\mathbb{K}^n) = \mathbb{K}^n \setminus \{0\} /_{\sim}, \quad v \sim w \iff v = \lambda w, \text{ for some } \lambda$$

PA = the set of lines through A = $= \pi(A)$, where $\pi : \mathbb{K}^n \to P$.

Sometimes instead of PA, simply A.

$\forall g \in GL(n, \mathbb{K}) \to$ defines a $g : P \to P$.

Same notation for both.

The ideal case allowing to construct F_2 in Γ:

· assume $\exists\, g \in \Gamma$ diagonalisable with:

– eigenvalues with multiplicity all in $\mathbb{R}_+$,

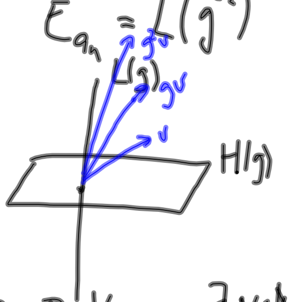
$$a_1 > a_2 \geq \dots \geq a_{n-1} > a_n > 0$$

– consider eigenspaces: $E_{a_1} = L(g)$

$E_{a_n} = L(g^{-1})$

$$H(g) = \bigoplus_{i=2}^{n} E_{a_i}$$

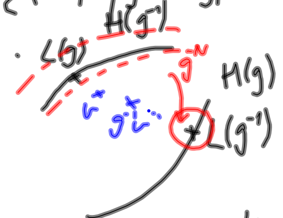
$$H(g^{-1}) = \bigoplus_{i=1}^{n-1} E_{a_i}$$



In P:

In P, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. g^N sends the complementary set of $\mathcal{N}_\varepsilon(H) = \{y \,;\, d(y,H) < \varepsilon\}$ inside $B(L(g), \varepsilon)$.

P



· assume $\exists\, h \in \Gamma$ with same properties. moreover $L(h) \notin H(g) \cup H(g^{-1})$

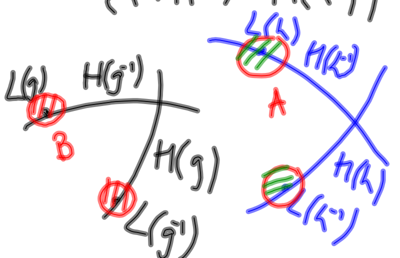
$L(h^{-1})$

similarly $L(g^{\pm 1}) \notin H(h) \cup H(h^{-1})$

Take $A = B(L(h),\varepsilon) \cup B(L(h^{-1}),\varepsilon)$

$B = B(L(g),\varepsilon) \cup B(L(g^{-1}),\varepsilon)$.

For ε small enough A disj. of $N_\varepsilon(H(g) \cup H(g^{-1}))$, B —"— $N_\varepsilon(H(h) \cup H(h^{-1}))$.



$\exists N \in \mathbb{N}$ s.t.

$g^N(A) \subseteq B$

$h^N(B) \subseteq A$

$\langle g^N, h^N \rangle \cong F_2$.

We try to approximate such a situation.

Standard = the Cartan decomposition =

$$GL(n,\mathbb{K}) = K \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} ; a_1 \geq a_2 \dots \geq a_n > 0 \right\} K$$

where $K = O(n,\mathbb{R})$ for $\mathbb{K} = \mathbb{R}$,

$U(n)$ for $\mathbb{K} = \mathbb{C}$.

i.e. $\forall M \in GL(n,\mathbb{K})$, $M = O \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix} O'$.

$O, O' \in K$.

Thm. in linear algebra

In $\mathbb{K}^n$, consider standard scalar prod:

$(x,y) = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$.

For every other scalar prod., $(\ ,\ )'$, $\exists$ a basis of $\mathbb{K}^n$ orthonormal for $(\ ,\ )$, orthogonal for $(\ ,\ )'$.

Ex. Apply to $(\ ,\ )' = (\ ,\ ) \circ M \times M$

$(x,y)' = (Mx, My)$

obtain $M = O \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} O'$

$a_1, \dots, a_n$ positive, not ordered.

To order use permutation matrices $O_\sigma = (e_{\sigma(1)} | \dots)$

Two cases in proof:

<u>Case 1</u> $\Gamma \leq GL(n, K)$ is unbounded (for $\|\cdot\|$) $\Rightarrow$ geometry.

<u>Case 2</u> Γ relatively compact in $GL(n, K)$ $\Rightarrow$ nb. theory.

<u>Case 1</u> Γ unbounded $\Rightarrow \exists$ a sequence $g_i = k_i \cdot d_i \cdot k_i'$ s.t.

$$d_i = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \text{ and } a_1(g_i) \longrightarrow \infty.$$

Fix (g_i) from now on.

Take m maximal s.t. $\dfrac{a_m(g_i)}{a_1(g_i)}$ has positive lim sup.

Up to taking subseq., assume

$$\lim \frac{a_m(g_i)}{a_1(g_i)} > 0.$$

We say that (g_i) is m-contracting.

$L = \mathrm{Span}\langle e_1, \dots, e_m\rangle$, $E = \mathrm{Span}\langle e_{m+1}, \dots, e_n\rangle$:

In $\mathbb{P}$:

$g_i = k_i d_i k_i'$, $k_i, k_i' \in K \Rightarrow$

$\Rightarrow$ up to taking a subseq, we may assume $k_i \to k$, $k_i' \to k'$.

<u>Lemma 1</u> TFAE:

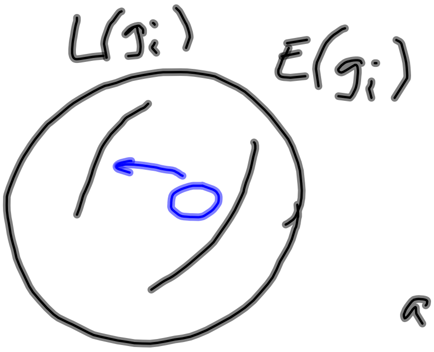
(1) g_i is 1-contracting;

(2) $\exists B$ ball in $\mathbb{P}$ s.t. $g_i|_B \to$ a constant transformation uniformly.

<u>Proof</u> We may assume $k_i = k$, $k_i' = k'$.

(1)

$(2) \Rightarrow (1)$: Assume, for contrad., that g_i is m-contracting, $m \geq 2$.



The limit of a ball will be a ball in $L(g_i)$

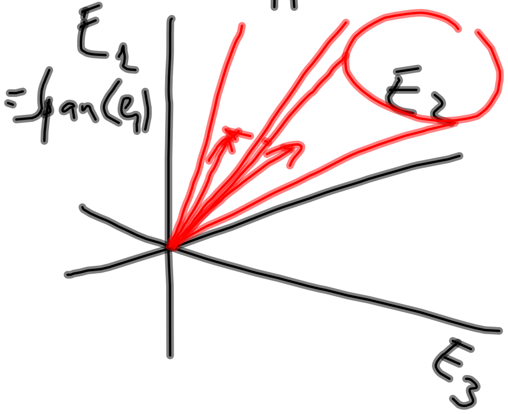
P

$m=2 \quad n=3$: $d_i = (a_{1i}, a_{2i}, a_{3i})$,

$a_{1i} \geq a_{2i} \geq a_{3i}$

$a_{1i} \to \infty$, $\frac{a_{2i}}{a_{1i}} \longrightarrow \ell > 0 \Rightarrow a_{1i} \geq a_{2i} \geq \frac{\ell}{2} a_{1i}$, for i large.

$\frac{a_{3i}}{a_{1i}} \longrightarrow 0$



$v = \sum v_i e_i \quad , \|v\| = 1$

$dv = a_1 v_1 e_1 + \dots$

$dv \sim v_1 e_1 + \frac{a_2}{a_1} v_2 e_2 + \frac{a_3}{a_1} v_3 e_3$

$i \to \infty \quad dv \to v_1 e_1 + \tilde{\ell} \, v_2 e_2$