Geometry of Nilpotent and Solvable Groups

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Lecture 9: Proof of Gromov's Theorem

We want to apply the Montgomery-Zippin Theorem, hence we want to represent our group Γ as a group of isometries of a metric space X that is :

- complete;
- connected, locally connected;
- proper (i.e. all balls are compact);
- of finite Hausdorff dimension.

We construct X as a limit of $(\Gamma, \frac{1}{\lambda_n}\mathrm{d})$, where d is a word metric and $\lambda_n \to \infty$.

We use an ultrafilter, i.e. a finitely additive measure $\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ with $\omega(\mathbb{N}) = 1$, to construct this limit.

Remark 1. If $\omega(A) = 1$ and $\omega(B) = 1$ then $\omega(A \cap B) = 1$.

Indeed, if $\omega(A \cap B) = 0$ then, as $A = (A \setminus B) \sqcup (A \cap B)$, it follows that $\omega(A \setminus B) = 1$.

Likewise $\omega(B \setminus A) = 1$ and since $A \setminus B$ and $B \setminus A$ are disjoint subsets it follows that $\omega(\mathbb{N}) \geq 2$, a contradiction.

Terminology: Let $P(n)$ be a proposition indexed by $n \in \mathbb{N}$. We say that '$P(n)$ is true ω-almost surely (ω-a.s.)' if $\omega(\{n \in \mathbb{N} \mid P(n) \text{ is true }\}) = 1$.

Recall that a non-principal ultrafilter is an ultrafilter such that $\omega(F) = 0$ for every $F \subseteq \mathbb{N}$ finite.

Zorn's Lemma (equivalent to the Axiom of choice) implies that non-principal ultrafilters exist.

Definition. *Let X be a non-empty set. Its ultrapower with respect to the ultrafilter ω, denoted X^ω, is the quotient of the set of sequences (x_n) in X with respect to the equivalence relation*

$$(x_n) =_\omega (y_n) \Leftrightarrow x_n = y_n \ \omega - \textit{almost surely.}$$

The equivalence class of the sequence (x_n) is denoted $(x_n)^\omega$.

If X has a structure (e.g. group, ring, order) then X^ω has the same structure.

For instance if X is a group then X^ω is a group with binary operation

$$(g_n)^\omega (h_n)^\omega = (g_n h_n)^\omega .$$

The space X has a copy in X^ω:

$$x \in X \mapsto (x)^\omega = \widehat{x} \in X^\omega .$$

For every $A \subseteq X$ we denote by $\widehat{A}$ its image by the above, i.e.

$$\widehat{A} = \{\hat{a} \mid a \in A\} .$$

Definition. *A subset $W \subseteq X^\omega$ is called internal if there exists a sequence of subsets (A_n) in X such that*

$$W = \{(x_n)^\omega \ ; \ x_n \in A_n\} .$$

We write $W = (A_n)^\omega$.

Proposition 2. *If $A \subseteq X$, A infinite, then $\widehat{A} \subseteq X^\omega$ cannot be internal.*

Proof. Assume $\widehat{A} = (B_n)^\omega$ for some sequence B_n of subsets.

For every $a \in A$, $\hat{a} \in (B_n)^\omega$, i.e.

$$a \in B_n \ \omega - \text{almost surely.} \tag{1}$$

Take an infinite sequence $a_1, a_2, ..., a_k, ...$ of distinct elements in A.

Let $I_k = \{n \in \mathbb{N} \mid n \geq k \, , \, \{a_1, a_2, ..., a_k\} \subseteq B_n\}$.

From (1) and Remark 1 it follows that $\omega(I_k) = 1$ for every k.

Note that $I_{k+1} \subseteq I_k$ and that $\bigcap_{n \geq 1} I_k = \emptyset$.

Define the sequence (x_n) such that $x_n = a_k$ for every $n \in I_k \setminus I_{k+1}$.

Since $\bigcap_{n\geq 1} I_k = \emptyset$, it follows that $I_1 = \bigcup_{k=1}^{\infty} (I_k \setminus I_{k+1})$. Thus the sequence (x_n) above is defined for all $n \in I_1$, and $\omega(I_1) = 1$. For all arguments in the ultrapower, the behaviour of a sequence on a set of indices of ω-measure zero does not matter.

By definition $x_n \in B_n$ for every $n \in I_1$, that is $x_n \in B_n$ ω-a.s.

Thus $(x_n)^\omega \in (B_n)^\omega = \widehat{A}$, hence $x_n = a$ ω-a.s. for some $a \in A$.

Let $J = \{n \in \mathbb{N} \mid x_n = a\}$, $\omega(J) = 1$. Remark 1 implies that $J \cap I_1 \neq \emptyset$, hence for some $k \in \mathbb{N}$, $J \cap (I_k \setminus I_{k+1}) \neq \emptyset$.

For $n \in J \cap (I_k \setminus I_{k+1})$ we have $x_n = a = a_k$.

The fact that $\omega(I_{k+1}) = 1$ and Remark 1 imply that $J \cap I_{k+1} \neq \emptyset$.

As $I_{k+1} = \bigcup_{j=k+1}^{\infty} (I_j \setminus I_{j+1})$ it follows that $J \cap (I_j \setminus I_{j+1}) \neq \emptyset$ for some $j \geq k+1$.

For an index n in the above intersection $x_n = a = a_j$. But as $j > k$ we have that $a_j \neq a_k$, so a contradiction. □

The following result is a consequence of Łoś' Theorem (see J. Bell and A. Slomson, *Models and Ultraproducts*, North-Holland, Amsterdam, 1969 ,or J. Keisler, *Foundations of Infinitesimal Calculus*, Prindel-Weber-Schmitt, Boston, 1976, Chapter 1).

Theorem (non-standard induction). *If a non-empty internal subset A^ω in $\mathbb{N}^\omega$ satisfies the properties:*

- $\widehat{1} \in A^\omega$;
- *for every* $n^\omega \in A^\omega$, $n^\omega + \widehat{1} \in A^\omega$;

then $A^\omega = \mathbb{N}^\omega$.

Definition. *A map $f^\omega : X^\omega \to Y^\omega$ is* internal *if there exists a sequence of maps $f_n : X_n \to Y_n$ such that $f^\omega((x_n)^\omega) = (f_n(x_n))^\omega$.*

If (X, d) is a metric space one can define a 'metric' d^ω on X^ω as the internal function defined by the constant sequence of functions (d), that is $\mathrm{d}^\omega : X^\omega \times X^\omega \to \mathbb{R}^\omega$,

$$\mathrm{d}^\omega\left((x_n)^\omega, (y_n)^\omega\right) = \left(\mathrm{d}(x_n, y_n)\right)^\omega . \tag{2}$$

The problem is that d^ω does not take values in $\mathbb{R}$ but in $\mathbb{R}^\omega$.

Limit spaces

We want to construct a limit of $\left(\mathrm{Cay}(\Gamma)\,,\,\frac{1}{\lambda_n}\mathrm{d}\right)$ when $\lambda_n \to \infty$.

Why not taking as "limit space" the space of sequences

$$\mathcal{S} = \{(x_n) \mid x_n \in \mathrm{Cay}(\Gamma)\}\,,$$

with the metric

$$D((x_n)\,,\,(y_n)) = \lim_\omega \frac{\mathrm{d}(x_n, y_n)}{\lambda_n}\ ?$$

Problem 1: This "distance" D may take the value ∞.

This is solved by restricting the space of sequences. Fix a sequence of basepoints $e = (e_n)$, and take

$$\mathcal{S}_e = \left\{ (x_n) \in \mathcal{S} \mid \left(\frac{d(x_n, e_n)}{\lambda_n}\right) \text{ bounded } \right\}.$$

Problem 2: We may have $(x_n) \neq (y_n) \in \mathcal{S}_e$ such that $\lim_\omega \frac{\mathrm{d}(x_n, y_n)}{\lambda_n} = 0$.

We solve this by considering the quotient $\mathcal{S}_e/\sim$, where

$$(x_n) \sim (y_n) \Leftrightarrow \lim_\omega \frac{\mathrm{d}(x_n, y_n)}{\lambda_n} = 0\,.$$

The quotient $\mathcal{S}_e/\sim$ is denoted by $\mathrm{Cone}_\omega(X; e, (\lambda_n))$ and called asymptotic cone of X with respect to ω, the sequence of basepoints e and the sequence of scaling constants (λ_n).

Convention: From now on all ultrafilters are non-principal, and we use the notation ω for such an ultrafilter.

Notation: We denote the equivalence class of a sequence (x_n) with respect to the equivalence relation $\sim$ by $\lim_\omega (x_n)$.

For a sequence of subsets $A_n \subseteq X$ we define the limit set $\lim_\omega (A_n) = \{\lim_\omega (a_n) \mid a_n \in A_n\,,\,\forall n \in \mathbb{N}\}$.

List of properties of asymptotic cones

1. Every $\text{Cone}_\omega(X; e, (\lambda_n))$ is complete.

2. X is geodesic $\Rightarrow$ every asymptotic cone is geodesic.

3. If G is a group then every $\text{Cone}_\omega(G; e, (\lambda_n))$ is isometric to $\text{Cone}_\omega(G; (1), (\lambda_n))$.

4. The subgroup G_1^ω of the ultrapower G^ω acts transitively on $\text{Cone}_\omega(G; (1), (\lambda_n))$, where
$$G_1^\omega = \left\{ (g_n)^\omega \; ; \; \left(\frac{|g_n|_S}{\lambda_n} \right) \text{ is bounded} \right\} .$$

Proof. (1) Let $(x^{(k)})$ be a Cauchy sequence in $\text{Cone}_\omega(X; e, (\lambda_n))$. It suffices to prove that a subsequence converges. We select a subsequence such that

$$D(x^{(k)}, x^{(k+1)}) < \frac{1}{2^k} \Leftrightarrow \lim_\omega \frac{\mathrm{d}\left(x_n^{(k)}, x_n^{(k+1)}\right)}{\lambda_n} < \frac{1}{2^k} \Leftrightarrow \mathrm{d}\left(x_n^{(k)}, x_n^{(k+1)}\right) < \frac{\lambda_n}{2^k} \; \omega - \text{a.s.}$$

Then we have $\omega(I_k) = 1$ for the set

$$I_k = \left\{ n \geq k \; ; \; \mathrm{d}\left(x_n^{(k)}, x_n^{(k+1)}\right) < \frac{\lambda_n}{2^k} \right\} .$$

We can assume that $I_{k+1} \subseteq I_k$, otherwise we replace I_{k+1} with $I_{k+1} \cap I_k$.

Thus we obtain a nested sequence of subsets I_k in $\mathbb{N}$ such that $\bigcap_{k \in \mathbb{N}} I_k = \emptyset$.

We define what we claim will be the 'limit point' of $(x^{(k)})$ as $\lim_\omega (y_n)$, with $y_n = x_n^{(k)}$ when $n \in I_k \setminus I_{k+1}$. The fact that $\bigcap_{n \geq 1} I_k = \emptyset$ implies that $I_1 = \bigcup_{k=1}^\infty (I_k \setminus I_{k+1})$, hence the above defines the sequence y_n for all $n \in I_1$. We have $\omega(I_1) = 1$ and, as for ultraproducts, in the arguments with asymptotic cones, the values of sequences on sets of indices of ω-measure zero do not matter.

For an arbitrary $k \in \mathbb{N}$ we prove that for all $n \in I_k$, $\frac{1}{\lambda_n}\mathrm{d}\left(x_n^{(k)}, y_n\right) < \frac{1}{2^{k-1}}$, i.e. ω-almost surely $\mathrm{d}\left(x_n^{(k)}, y_n\right) < \frac{1}{2^{k-1}}$; this implies that $D\left(x^{(k)}, y\right) \leq \frac{1}{2^{k-1}}$.

For every $n \in I_k = \bigcup_{j=k}^\infty (I_j \setminus I_{j+1})$ there exists $j \geq k$ such that $n \in I_j \setminus I_{j+1}$. By definition $y_n = x_n^{(j)}$.

Since $n \in I_j \subseteq I_{j-1} \subseteq \cdots \subseteq I_{k+1} \subseteq I_k$ we may write

$$\frac{\mathrm{d}\left(x_n^{(k)}, x_n^{(j)}\right)}{\lambda_n} \leq \frac{\mathrm{d}\left(x_n^{(k)}, x_n^{(k+1)}\right)}{\lambda_n} + \cdots + \frac{\mathrm{d}\left(x_n^{(j-1)}, x_n^{(j)}\right)}{\lambda_n} \leq$$

$$\frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{j-1}} \leq \frac{1}{2^k} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{k-1}} .$$

Thus we have $D\left(x^{(k)}, y\right) \leq \frac{1}{2^{k-1}}$, hence $x^{(k)} \to y$.

(2) Given two points $\lim_\omega (x_n)$ and $\lim_\omega (y_n)$ take geodesics $[x_n, y_n]$. Their limit set $\lim_\omega ([x_n, y_n])$ is a geodesic joining $\lim_\omega (x_n)$ and $\lim_\omega (y_n)$.

(3) The map $\mathrm{Cone}_\omega(G; e, (\lambda_n)) \to \mathrm{Cone}_\omega(G; (1), (\lambda_n))$ defined by $\lim_\omega (x_n) \mapsto \lim_\omega (e_n^{-1} x_n)$ is an isometry. □

From the above it follows that every asymptotic cone of a Cayley graph of a group is complete and geodesic (hence connected and locally connected).

Our next goal is to prove the implication: 'Γ of polynomial growth $\Rightarrow$ one asymptotic cone of Γ is proper and of finite Hausdorff dimension.'

Theorem (Hopf-Rinow Theorem). *If (X, d) is a complete, geodesic, locally compact metric space then it is proper.*

Thus, instead of properness, it suffices to prove local compactness for asymptotic cones.

Choice of the scaling sequence (λ_n)

Proposition 3. *Assume that there exists $R = (R_n)^\omega$ in the ultrapower $\mathbb{R}_+^\omega$ such that the growth function satisfies:*

$$\mathfrak{G}_\Gamma(R_n) = \mathrm{card}\, B_\Gamma(1, R_n) \leq C R_n^a ,\ \forall n \in \mathbb{N} ,$$

where $C > 0$ and $a \in \mathbb{N}$ are constants independent of n.

Then there exists $\lambda \in [\log R\,,\, R] \subset \mathbb{R}_+^\omega$ such that the ball $B\left(1, \frac{\lambda}{4}\right)$ in the ultrapower Γ^ω endowed with the metric defined in (2) satisfies the following. For every $i \in \mathbb{N}$, $i \geq 4$, all the sets of $\frac{\lambda}{i}$-separated points in the ball $B\left(1, \frac{\lambda}{4}\right)$ have cardinality at most i^{a+1}.

A subset A is ε-separated if for every $a_1, a_2 \in A$, $\mathrm{d}(a_1, a_2) \geq \varepsilon$.

Proof. Assume that the conclusion of the proposition is false, i.e. for every $\lambda \in [\log R\,,\,R] \subset \mathbb{R}_+^\omega$ there exists $i \in \mathbb{N}$, $i \geq 4$, such that the ball $B\left(1, \frac{\lambda}{4}\right)$ contains at least i^{a+1} points that are $\frac{\lambda}{i}$-separated.

Define the map $F : [\log R\,,\,R] \to \mathbb{N} \hookrightarrow \mathbb{N}^\omega$, $F(\lambda) =$ the minimal $i \in \mathbb{N}$, $i \geq 4$, with the above property.

- F is an internal map defined by the sequence of maps:

  $F_n : [\log R_n\,,\,R_n] \to \mathbb{N}$, $F_n(x) =$ the minimal $i \in \mathbb{N}$, $i \geq 4$, such that $B_\Gamma\left(1\,,\,\frac{x}{4}\right)$ contains at least i^{a+1} points that are $\frac{x}{i}$-separated.

- the image of F is therefore internal.

On the other hand, by definition, the image of F is contained in $\widehat{\mathbb{N}} \subseteq \mathbb{N}^\omega$, therefore it equals $\widehat{A}$ for some $A \subseteq \mathbb{N}$.

Proposition 2 implies that A must be finite.

Thus F takes values in $\{4, ..., N\}$ for some integer $N \in \mathbb{N}$.

We have obtained that for every $\lambda \in [\log R, R]$ there exists $i \in \{4, ..., N\}$ such that the ball $B\left(1, \frac{\lambda}{2}\right)$ contains at least i^{a+1} disjoint balls of radii $\frac{\lambda}{2i}$.

For R there exists $i_1 \in \{4, ..., N\}$ such that the ball $B\left(1, \frac{R}{2}\right)$ contains at least i_1^{a+1} disjoint balls

$$B\left(x_1(1), \frac{R}{2i_1}\right)\,,\, B\left(x_2(1), \frac{R}{2i_1}\right)\,,\,\dots\,,\, B\left(x_{t_1}(1), \frac{R}{2i_1}\right) \text{ with } t_1 \geq i_1^{a+1}\,.$$

All the balls in the list above are isometric to $B\left(1, \frac{R}{2i_1}\right)$. Clearly $\frac{R}{i_1} \in [\log R\,,\,R]$, hence there exists $i_2 = F\left(\frac{R}{i_1}\right)$ such that the ball $B\left(1, \frac{R}{2i_1}\right)$ contains at least i_2^{a+1} disjoint balls of radii $\frac{R}{2i_1 i_2}$.

It follows that $B\left(1, \frac{R}{2}\right)$ contains a family of disjoint balls

$$B\left(x_1(2), \frac{R}{2i_1 i_2}\right)\,,\, B\left(x_2(2), \frac{R}{2i_1 i_2}\right)\,,\,\dots\,,\, B\left(x_{t_2}(2), \frac{R}{2i_1 i_2}\right) \text{ with } t_2 \geq i_1^{a+1} i_2^{a+1}\,.$$

We started a non-standard induction. We continue, and find $u \in \mathbb{N}^\omega$ such that $B\left(1, \frac{R}{2}\right)$ contains a family of disjoint balls

$$B\left(x_1(u), \frac{R}{2i_1 i_2 ... i_u}\right)\,,\, B\left(x_2(u), \frac{R}{2i_1 i_2 ... i_u}\right)\,,\,\dots\,,\, B\left(x_{t_u}(u), \frac{R}{2i_1 i_2 ... i_u}\right)\,,$$

with $t_u \geq (i_1 i_2 ... i_u)^{a+1}$.

The process stops for $u \in \mathbb{N}^\omega$ such that

$$\frac{R}{i_1 i_2 ... i_u} < \log R \leq \frac{R}{i_1 i_2 ... i_{u-1}} \leq \frac{NR}{i_1 i_2 ... i_u} \Leftrightarrow$$

$$\frac{R}{\log R} < i_1 i_2 ... i_u \leq \frac{NR}{\log R}.$$

We obtained that the ball $B\left(1, \frac{R}{2}\right)$ in $(X^\omega, \mathrm{d}_\omega)$ contains at least $(i_1 i_2 ... i_u)^{a+1}$ elements, hence at least $\left(\frac{R}{\log R}\right)^{a+1}$ elements. This implies that ω-almost surely $B(1, R_n)$ contains at least $\left(\frac{R_n}{\log R_n}\right)^{a+1}$ elements.

But by hypothesis $B\left(1, \frac{R_n}{4}\right)$ contains at most CR_n^a elements, hence $\frac{R_n}{(\log R_n)^{a+1}} \leq C$, a contradiction. □

Now take $\lambda = (\lambda_n)$ as in Proposition 3, and $X = \mathrm{Cone}_\omega(\Gamma; 1, \lambda)$.

In X the ball $B\left(1, \frac{1}{4}\right)$ contains, for every $i \in \mathbb{N}$, $i \geq 4$, at most i^{a+1} points that are $\frac{1}{i}$-separated.

Lemma 4. *The ball $B\left(1, \frac{1}{4}\right)$ in X is compact.*

Proof. For every open cover $\{U_j \mid j \in J\}$, take $\varepsilon > 0$ such that every ball of radius ε is contained in some U_j (i.e. ε is the Lebesgue number of the cover).

Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

$B\left(1, \frac{1}{4}\right)$ contains at most n^{a+1} points that are $\frac{1}{n}$-separated.

Take a maximal $\frac{1}{n}$-separated set, $\{x_1, ..., x_N\}$ with $N \leq n^{a+1}$. For every $r \in \{1, 2, ..., n\}$, the ball $B\left(x_r, \frac{1}{n}\right)$ is contained in some U_{j_r}.

We have

$$B\left(1, \frac{1}{4}\right) \subseteq \bigcup_{r=1}^{N} B\left(x_r, \frac{1}{n}\right) \subseteq \bigcup_{r=1}^{N} U_{j_r}.$$

The first inclusion is due to the maximality of the $\frac{1}{n}$-separated set $\{x_1, ..., x_N\}$ (implying that any other point must be at distance $< \frac{1}{n}$ from one of the points x_r. □

Lemma 4 and the fact that X is homogeneous (see page 4, property (4)) imply that X is locally compact.

We must prove that X has finite Hausdorff dimension. The following well-known result allows to simplify this problem further.

Proposition 5. *For a proper metric space X such that $Isom(X)$ acts transitively on it, it suffices to prove that some small ball B has finite Hausdorff dimension.*

This is because the hypotheses imply that X is covered by countably many translates gB of B by elements g in $Isom(X)$.

We prove that the closed ball $B = B'\left(1, \frac{1}{8}\right) \subset B\left(1, \frac{1}{4}\right)$ has finite Hausdorff dimension.

A compact metric space K has Hausdorff dimension $< \beta$ if there exists a sequence of covers of K by balls $\{B_i(n) \; ; \; i \in I_n\}$ such that:

- $\max\{\text{radius}(B_i(n)) \; ; \; i \in I_n\}$ converges to 0 as $n \to \infty$;
- $\sum_{i \in I_n} [\text{radius}(B_i(n))]^\beta$ converges to 0 as $n \to \infty$.

Remark. *Since K is compact, we may assume that all the covers by balls above are finite.*

For $K = B = B'\left(1, \frac{1}{8}\right)$ take, for every $n \in \mathbb{N}$, a maximal $\frac{1}{n}$-separated subset, $x_1(n), ..., x_{k_n}(n)$.

We know that $k_n \leq n^{a+1}$.

The n-th cover of B is

$$B\left(x_1(n)\, , \, \frac{1}{n}\right), ..., B\left(x_{k_n}(n)\, , \, \frac{1}{n}\right) .$$

- radii are $\frac{1}{n} \to 0$;
- $\sum_{i=1}^{k_n} \frac{1}{n^\beta} \leq \frac{n^{a+1}}{n^\beta}$ converges to 0 if $\beta > a+1$.

Thus B has finite Hausdorff dimension, hence so does X.

We can now apply Montgomery-Zippin to the group $H = Isom(X)$.

We have

$$\begin{array}{ccccc} \Gamma & \hookrightarrow & \Gamma_1^\omega & \xrightarrow{\varphi} & Isom(X)\, , \\ \gamma & \mapsto & (\gamma)^\omega\, . & & \end{array}$$

We obtain a homomorphism $\varphi : \Gamma \to Isom(X)$.

$\varphi(\Gamma) \leq Isom(X)$, and $\varphi(\Gamma)$ has polynomial growth because Γ has polynomial growth.

By the argument in last lecture it follows that $\varphi(\Gamma)$ is virtually nilpotent.

In order to prove Gromov's Theorem, we argue by induction on the degree of the polynomial growth, i.e. on $a \in \mathbb{N}$ such that $\mathfrak{G}_\Gamma(n) \preceq n^a$.

$a = 0 \Rightarrow \Gamma$ is finite.

Assume that Gromov's Theorem is true for a and consider Γ with $\mathfrak{G}_\Gamma(n) \preceq n^{a+1}$.

Case 1. $\varphi(\Gamma)$ is infinite.

In that case the abelianization $\varphi(\Gamma)_{ab}$ (or that of a finite index subgroup) is infinite. Hence, up to replacing Γ by a finite index subgroup, we may assume that there exists a surjective homomorphism $\varphi(\Gamma) \to \mathbb{Z}$.

Then we have a short exact sequence

$$1 \to N \to \Gamma \to \mathbb{Z} \to 1\,. \tag{3}$$

Lemma 6. *Suppose that Γ is a finitely generated group such that $\mathfrak{G}_\Gamma(n) \preceq n^{a+1}$, and Γ fits into a short exact sequence as in (3). Then N is finitely generated and $\mathfrak{G}_N(n) \preceq n^a$.*

Proof. Let $\{s_1, ..., s_k\}$ be a set of generators of Γ, and let $\gamma \in \Gamma$ be an element which projects to the generator 1 of $\mathbb{Z}$.

For each i there exists $m_i \in \mathbb{Z}$ such that $\pi(s_i \gamma^{m_i}) = 0 \in \mathbb{Z}$.

Define elements $g_i := s_i \gamma^{m_i}$, $i = 1, ..., k$. The set $\{g_1, ..., g_k, \gamma\}$ generates Γ, and $g_1, ..., g_k$ are in N.
The (infinite) subset C of N defined by

$$C := \{\gamma_{m,i} := \gamma^m g_i \gamma^{-m}\ ;\ m \in \mathbb{Z}, i = 1, ..., k\}$$

generates N.

To see this it suffices to write an element in N as a word in $\{g_1, ..., g_k, \gamma\}$, re-write it as a product of conjugates in C, plus some power γ^n, and deduce by projecting in $\mathbb{Z}$ that $n = 0$.

Lemma 1 in Lecture 5 (page 1) implies that N is generated by a finite subset F of C.

Note that Γ is isomorphic to $N \rtimes \mathbb{Z}$, hence, if we consider N with the generating set F and Γ with the generating set $F \cup \{\gamma^{\pm 1}\}$, we obtain that $n\mathfrak{G}_N(n) \leq \mathfrak{G}_\Gamma(2n) \preceq n^{a+1}$, whence $\mathfrak{G}_N(n) \preceq n^a$. □

We may then use the induction hypothesis to conclude that N is virtually nilpotent, hence Γ is virtually polycyclic, which, by Wolf's Theorem, implies that Γ is virtually nilpotent.

Case 2. $\varphi(\Gamma)$ is finite. Up to finite index, we may assume that $\varphi(\Gamma) = \{\mathrm{id}\}$.

If $\Gamma = \mathbb{Z}^n$ this is indeed what occurs: for every γ, the pairs of sequences $\gamma(x_n) = (x_n\gamma)$ and (x_n) clearly satisfy $\mathrm{d}(x_n\gamma, x_n) \leq |\gamma|$.

We define the following displacement functions. For every $\gamma \in \Gamma$, $x \in \Gamma$ and $r > 0$ we define

$$\Delta(\gamma, x, r) = \max\{\mathrm{d}(y, \gamma y) \; ; \; y \in B(x, r)\} \, .$$

When $x = 1$ we write $\Delta(\gamma, r)$ for the displacement function.

Let S be a finite generating set of Γ. Define $\Delta(S, x, r) = \max_{s \in S} \Delta(s, x, r)$.

Likewise we write $\Delta(S, r)$ when $x = 1$.

Lemma. *If the function $r \mapsto \Delta(S, r)$ is bounded then Γ is virtually abelian.*

Proof. Assume $\Delta(S, r) \leq C$ for every $r \geq 0$, where C is a constant uniform in r.

For a fixed $s \in S$ and every $x \in \Gamma$,

$$\mathrm{d}(sx, x) \leq C \Leftrightarrow |x^{-1}sx|_S \leq C \, .$$

It follows that s has finitely many conjugates.

Consider the action of Γ on itself by conjugation.

The orbit map of s

$$\Gamma \to \Gamma \, , \; x \mapsto x^{-1}sx \, ,$$

has its image in the ball $B(1, C)$.

The stabilizer of s by this action is the centralizer $Z_\Gamma(s)$. It follows that $Z_\Gamma(s)$ has finite index in Γ.

The intersection $\bigcap_{s \in S} Z_\Gamma(s) = Z(\Gamma)$ likewise has finite index in Γ, and it is obviously abelian. ∎

Assume then that the function $r \mapsto \Delta(S, r)$ is unbounded.

Lemma. *For every $\varepsilon > 0$ there exists a sequence (x_n) in Γ such that*

$$\lim_\omega \frac{\max_{s\in S}\Delta(x_n s x_n^{-1}, \lambda_n)}{\lambda_n} = \varepsilon\,. \tag{4}$$

Proof. Since φ has trivial image it follows that $\lim_\omega \frac{\Delta(S,\lambda_n)}{\lambda_n} = 0$. In particular $\Delta(S, \lambda_n) = \Delta(S, 1, \lambda_n)$ is at most $\frac{\varepsilon}{2}\lambda_n$ ω-almost surely.

On the other hand, since $\Delta(S, r)$ is unbounded, for every n there exists a point p_n such that $2\varepsilon\lambda_n \le \max_{s\in S} \mathrm{d}(sp_n, p_n) \le \Delta(S, p_n, \lambda_n)$.

It is easy to check that for a fixed λ, the function $p \mapsto \Delta(S, p, \lambda)$ is 2-Lipschitz. This continuity and the considerations above imply that ω-almost surely there exists a point x_n such that $\Delta(S, x_n, \lambda_n) = \varepsilon\lambda_n$. ☐

For a sequence (x_n) as in the previous lemma we define a new homomorphism

$$\varphi_\varepsilon : \Gamma \to Isom(X),\ \varphi_\varepsilon(\gamma) = (x_n\gamma x_n^{-1})^\omega\,.$$

Clearly the image of φ_ε is not $\{\mathrm{id}\}$. If the image of φ_ε is infinite, we argue as before.

Assume that the image of φ_ε is finite, for every $\varepsilon > 0$. Note that by construction, for every $s \in S$, $\varphi_\varepsilon(s)$ has maximal displacement in the ball $B(1, 1) \subseteq X$ at most ε.

This means that, as $\varepsilon \to 0$, the elements $\varphi_\varepsilon(s)$ are in smaller and smaller neighbourhoods of the identity element in the topological group $H = Isom(X)$ (endowed with the compact-open topology.)

The group H is a Lie group, we have denoted by H_0 its connected component of the identity, and two properties of Lie groups will allow to finish the argument.

Theorem. 1. *Every finite subgroup of H_0 contains an abelian subgroup of index at most $I = I(H_0)$.*

2. *For every $m \in \mathbb{N}$ there exists a neighbourhood of the identity element in H_0 that does not contain cyclic subgroups of order m.*

The first statement in the above Theorem implies that, by eventually replacing Γ with a finite index subgroup, we may assume that all $\varphi_\varepsilon(\Gamma)$ are abelian.

If the order of $\varphi_\varepsilon(\Gamma)$ is bounded by a constant M uniform in ε, this implies that for every $\gamma \in \Gamma$, $\varphi_\varepsilon(\Gamma)$ has maximal displacement in the ball $B(1, 1) \subseteq X$ at most

$M\varepsilon$. Thus $\varphi_\varepsilon(\Gamma)$ is in smaller and smaller neighbourhoods of the identity element in H_0.

Consequently, smaller and smaller neighbourhoods of the identity element in H_0 contain cyclic subgroups of fixed order, contradicting the second part of the above Theorem.

It follows that for some $\varepsilon_n \to 0$, the orders of $\varphi_{\varepsilon_n}(\Gamma)$ diverge to infinity.

All $\varphi_{\varepsilon_n}(\Gamma)$ are abelian, therefore they are quotients of the abelianization Γ_{ab} of Γ. It follows that the abelianization of Γ is infinite, hence we may define a surjective homomorphism $\Gamma \to \Gamma_{ab} \to \mathbb{Z}$.

Lemma 6 and the inductive hypothesis allow to finish the argument. $\square$