Exercise 1. Consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \; ; \; a = 2^n \, , \, b = \frac{m}{2^k} \, , \, n, m, k \in \mathbb{Z} \right\} .$$

Note that it has a finite generating set S composed of $d = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and their inverses.

(1) Find the conjugate $d^{-1}ud$.

(2) Prove that for every $k \in \mathbb{N}$, $|u^k|_S \preceq \ln k$.

(3) Prove that the group G has *exponential growth*, that is the growth function

$$\mathfrak{G}_S^G(n) = \operatorname{card} B(1, n) = \operatorname{card} \{ g \in G \; ; \; \operatorname{dist}_S(1, g) < n \}$$

satisfies $\mathfrak{G}_S^G(n) \asymp e^n$.

Exercise 2. Let G be a finitely generated nilpotent group. Prove that for every subgroup H in G, there exists an integer $m = m(H, G)$ such that for every two generating sets X of H, and S of G respectively, and the corresponding word metrics dist_X on H and dist_S on G, we have that

$$\frac{1}{L} \operatorname{dist}_S(h, h') \leq \operatorname{dist}_X(h, h') \leq L \left[\operatorname{dist}_S(h, h') \right]^m \, , \, \forall h, h' \in H \, ,$$

where L is a constant depending on the choice of X and S.

[*Hint.* Use an induction on the nilpotency class of G. The statement for G abelian group may be assumed without proof. In that case it is well known that all $m(H, G)$ may be taken equal to 1.]