$$I = Ax_1 + .. + Ax_n$$

$$A = \mathbb{K}[X_1, \dots, X_n]$$

① $$z \longmapsto I_z = Ap_1 + \dots + Ap_k$$

$$p_1 = 0, \dots, p_k = 0$$

② $$I_{z_1} \subseteq I_{z_2} \subseteq \dots \subseteq I_{z_n} \subseteq \dots$$

$Z_i, i \in I, \bigcap_{i \in I} Z_i$ defined by

$\bigcup I_{Z_i} = \{ P \ ; \ P|_{Z_i} = 0 \ ; \text{ for some } i \}.$

$Z_1 \cup \ldots \cup Z_k$

$\forall i, \exists$ a finite set of polynomials defining Z_i, $F_i \subseteq \mathbb{K}[X_1, \ldots, X_n]$.

$Z_1 \cup \ldots \cup Z_k$ defined by

$P_1 \cdot \ldots \cdot P_k = 0, \ P_i \in F_i.$

X compact: $\forall$ family of closed sets $\{Z_i;\ i\in I\}$, $\bigcap_{i\in I} Z_i = \phi \Rightarrow$

$\Rightarrow \exists\, F\subseteq I$ finite s.t. $\bigcap_{i\in F} Z_i = \phi$.

Assume $\forall F\subseteq I$ finite, $\bigcap_{i\in F} Z_i \neq \phi$.

We produce a DC that does not stabilize:

Pick $Z_{i_0}\neq\phi$, $\exists\, i_1\in I$ s.t.

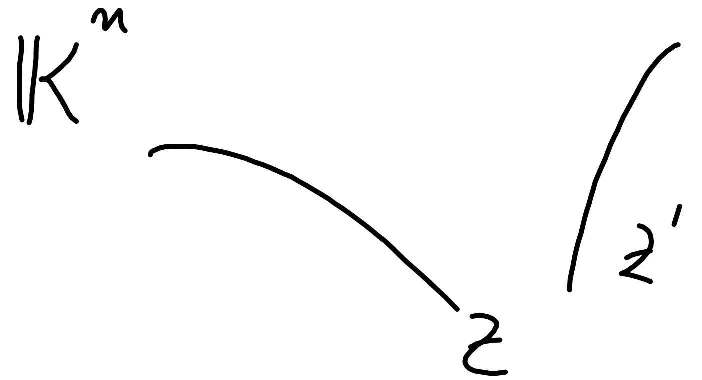
$$Z_{i_1}\cap Z_{i_0} \subsetneq Z_{i_0}.$$

Assume we found

$$Z_{i_0} \supsetneq Z_{i_0}\cap Z_{i_1} \supsetneq \dots \supsetneq \underbrace{Z_{i_0}\cap\dots\cap Z_{i_m}}$$

$\exists\, i_{m+1}$ s.t.

$$Z_{i_0}\cap\dots\cap Z_{i_m}\cap Z_{i_{m+1}} \subsetneq$$



$\forall$ pair of open sets in A :

$\emptyset \neq A \cap U,\ A \cap U',\ U, U'$ open in X

$U \cap \overline{Y},\ U' \cap \overline{Y} \neq \emptyset \Rightarrow U \cap Y,\ U' \cap Y \neq \emptyset$

$\Rightarrow U \cap U' \cap Y \neq \emptyset$

Conn. case

Proof of theorem

$$\mathcal{F} = \{ Y \subseteq X \text{ closed} ;\ Y \neq \text{finite union of maximal irred. subsets} \} \neq \emptyset$$

We had the DCC for closed sets.

$\exists\, Y$ minimal in $\mathcal{F}$.

We can write Y as union of proper closed subsets:

$$Y = Z_1 \cup \dots \cup Z_k ,\ Z_i \notin \mathcal{F} .$$

$$Z_i = \bigcup_{\substack{j \in F_i \\ \text{f}}} M_j$$

$$Y = \bigcup_{i=1} \bigcup M_j$$

$$X_i \subset \bigcup_{j \neq i} X_j \Rightarrow X_i = \bigcup_{j \neq i} (X_j \cap X_i)$$

One $X_j \cap X_i = X_i$

$$X = \bigcup_{i=1}^{n} X_i = \bigcup_{j=1}^{m} Y_j$$

$$\forall j, \exists i_j \text{ s.t. } Y_j = X_{i_j}$$

$$\exists \{1, \dots, m\} \to \{1, \dots, n\} \text{ inj.}$$

$\exists k$ not in image :

$$X_k \subseteq \bigcup Y_j \subseteq \bigcup_{i \neq k} X_i \text{ ※}$$

Quadratic form $q: \mathbb{K}^n \to \mathbb{K}$

$$q(x) = B(x,x)$$

$B: \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ bilinear, symmetric

$$B(x,y) = \langle Ax, y \rangle, \quad A^T = A.$$

$$O(q): q \circ M = q$$

Proof $\quad b : SL(V) \times SL(V) \to SL(V)$

$$(A, B) \longrightarrow A \cdot B$$

b cont. wrto Zarisky top.

$$b(\bar{\Gamma} \times \bar{\Gamma}) \subseteq \overline{b(\Gamma \times \Gamma)} = \bar{\Gamma}$$

Likewise for inversion.

<u>Ex. of semisimple</u>

$$S = SL(n,\mathbb{K}) / \{\pm I_n\}$$

$$G = GL(n,\mathbb{R}), \quad Rad(G) = \{\lambda I_n\}$$

$$G = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right), \quad Rad(G) = \left(\begin{array}{c|c} \lambda I_k & * \\ \hline 0 & \lambda' I_m \end{array}\right)$$

$$G = Rad(G) \rtimes \left(\begin{array}{c|c} SL(k,\mathbb{K}) & 0 \\ \hline 0 & SL(m,\mathbb{K}) \end{array}\right)$$

$$\rho : \underbrace{\text{Semisimple}}_{S} \longrightarrow GL(V)$$

$$V = V_1 \oplus \ldots \oplus V_k$$

$$\rho(S)(V_i) \subseteq V_i$$

$$\nexists\, W \subsetneq V_i \text{ s.t. } \rho(S)(W) \subseteq W.$$

$$\langle x, y \rangle = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$$

$\exists$ an orthonormal basis $B = \{u_1, \dots, u_n\}$

$$M^B(g) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

$$a_i = B(u_i, u_i) \geq 0$$

Take $\langle x, y\rangle = \sum x_i \bar{y_i}$

Take the bilinear form

$$b(x,y) = \langle\ ,\ \rangle \circ (M \times M)$$
$$= \langle Mx, My\rangle = \langle M^T M x, y\rangle.$$

Take $\mathcal{B} = \{u_1, \dots, u_n\}$ orthon.

$\mathcal{B}_0 = \{e_1, \dots, e_n\} \xrightarrow[P]{} \mathcal{B}$

$$P = \left(u_1 \middle| \dots \middle| u_n\right) \Rightarrow P^T P = I$$

$$P^T M^T M P = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = D^{-2}$$

$$a_i = b(u_i, u_i) \qquad D^{-1} = \begin{pmatrix} \frac{1}{\sqrt{a_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{a_n}} \end{pmatrix}$$

$$(MPD)^T MPD = I \Rightarrow MPD = A \Rightarrow M = AD^{-1}P^T$$

$\Rightarrow$

Proof $\mathbb{K}=\mathbb{R}$, $U, V \in O(n)$

Assume $M = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$. $\|M\| \le a_1$

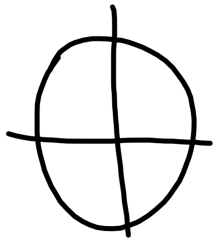
i.e. $\|Mv\| \le a_1 \|v\|$

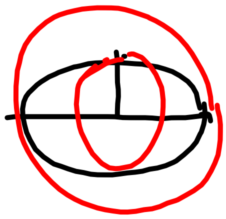
M acts on $\mathbb{K}^n \wedge \mathbb{K}^n$:

eigenvectors $e_i \wedge e_j$, eigenval. $a_i a_j$.

$$\|Mu \wedge Mv\| \le a_1^2 \|u \wedge v\|$$

$$\|Mu\| \ge a_n \|u\|$$





$$d(Mu, Mv) = \frac{\|Mu \wedge Mv\|}{\|Mu\| \cdot \|Mv\|}$$

$$\le \frac{a_1^2 \|u \wedge v\|}{a_n^2 \|u\| \cdot \|v\|}$$

$$\mathbb{P}\mathbb{R}^n = S^{n-1}/_{\text{opp.}}$$