

Amenable groups, Jacques Tits' Alternative Theorem

Cornelia Druţu

Oxford

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Last lecture

- **Paradoxical** metric spaces (in particular groups)= spaces piecewise congruent with several copies of themselves.
- **Tarski number of a paradoxical space**= minimal number of subsets in a paradoxical decomposition.
- We proved that F_2 , the free group of rank 2, is **paradoxical** with Tarski number 4.
- We remarked that $SO(3)$ (and $SO(n)$, $n \geq 3$) contains copies of F_2 .
- We deduced from the above and the **Axiom of Choice** that the unit ball in \mathbb{R}^n , $n \geq 3$, is paradoxical.
- **R. M. Robinson**: the Tarski number for the unit ball in \mathbb{R}^n , $n \geq 3$, is five (proof in notes).

This inspired J. von Neumann to define **amenability**.

For groups, this property is **the negation of being paradoxical**.

The initial definition of von Neumann (for groups) was in terms of **invariant means**.

Here we begin with equivalent **metric** definitions for **graphs**, then move on to **groups** and **means**.

Convention: In what follows all graphs \mathcal{G} are **connected, unoriented**, and have **bounded geometry**: valency of vertices uniformly bounded.

All their edges have length 1.

adjacent vertices = endpoints of one edge.

Cheeger constant

$F \subset V = V(\mathcal{G})$ set of vertices in a graph \mathcal{G} .

vertex-boundary of F , $\partial_V F$ = set of vertices in $V \setminus F$ adjacent to vertices in F .

Cheeger constant or **Expansion Ratio** of \mathcal{G} :

$$h(\mathcal{G}) = \inf \left\{ \frac{|\partial_V F|}{|F|} : F \text{ finite non-empty subset of } V \right\}.$$

amenable graph = Cheeger constant zero.

Equivalently, $\exists F_n$ non-empty finite in V such that

$$\lim_{n \rightarrow \infty} \frac{|\partial_V F_n|}{|F_n|} = 0.$$

(F_n) = Følner sequence for the graph.

non-amenable graph = positive Cheeger constant or empty graph.

Finite graphs are amenable: take $F_n = V$.

Notation

Let (X, dist) , F subset of X and $C > 0$:

$$\overline{\mathcal{N}}_C(F) = \{x \in X : \text{dist}(x, F) \leq C\}, \quad \mathcal{N}_C(F) = \{x \in X : \text{dist}(x, F) < C\}.$$

$\mathcal{B}(X) :=$ **bounded perturbations of the identity**, i.e. maps $f : X \rightarrow X$ such that

$$\text{dist}(f, \text{id}_X) = \sup_{x \in X} \text{dist}(f(x), x) \text{ is finite.}$$

Lemma

*In a group with a word metric, $\mathcal{B}(G)$ consists of **piecewise right translations**: given $f \in \mathcal{B}(G)$ there exist h_1, \dots, h_n in G and a decomposition $G = T_1 \sqcup T_1 \sqcup \dots \sqcup T_n$ such that f restricted to T_i coincides with $R_{h_i}(x) = xh_i$.*

TFAE:

- (a) \mathcal{G} is non-amenable.
- (b) **expansion condition**: $\exists C > 0$ such that for every finite $F \subset V$,
 $|\overline{\mathcal{N}}_C(F)| \geq 2|F|$.
- (c) $\exists f \in \mathcal{B}(V)$ such that $\forall v \in V$, $f^{-1}(v)$ contains exactly two elements.
- (d) **(Gromov's condition)** $\exists f \in \mathcal{B}(V)$ such that $\forall v \in V$, $f^{-1}(v)$ contains at least two elements.

Consequence: the Cayley graph of F_2 with respect to $S = \{a^{\pm 1}, b^{\pm 1}\}$ is non-amenable.

Slight variation

Remark

Property (b) can be replaced by (b'): for some $\beta > 1$ there exists $C > 0$ such that $|\overline{\mathcal{N}}_C(F) \cap V| \geq \beta|F|$. Indeed

$$\forall F, |\overline{\mathcal{N}}_C(F)| \geq \alpha|F| \Rightarrow \forall k \in \mathbb{N}, |\overline{\mathcal{N}}_{kC}(F)| \geq \alpha^k|F|.$$

Reminder Graph theory

Bipartite graph = vertex set $V = Y \sqcup Z$, edges with one endpoint in X , one in Y .

Given two integers $k, l \geq 1$, a **perfect (k, l) -matching** = a subset of edges such that each vertex in Y is the endpoint of exactly k edges in M , while each vertex in Z is the endpoint of exactly l edges in M .

Theorem (Hall-Rado matching theorem)

A bipartite graph of bounded geometry such that:

- *For every finite subset $A \subset Y$, its vertex-boundary $\partial_V A$ contains at least $k|A|$ elements.*
- *For every finite subset B in Z , its vertex-boundary $\partial_V B$ contains at least $|B|$ elements.*

has a perfect $(k, 1)$ -matching.

Amenability and growth

A **growth function** of a graph \mathcal{G} with a **basepoint** $x \in V$

$$\mathfrak{G}_{\mathcal{G},x}(R) := |\bar{B}(x, R) \cap V|,$$

where $\bar{B}(x, R)$ is the closed R -ball centered at x .

Dependence on the choice of x up to asymptotic equivalence.

asymptotic inequality between $f, g : X \rightarrow \mathbb{R}$ with $X \subset \mathbb{R}$:

$f \preceq g$ if there exist $a, b > 0$ such that $f(x) \leq ag(bx)$ for every $x \in X$, $x \geq x_0$ for some fixed x_0 .

f and g are **asymptotically equal** ($f \asymp g$) if $f \preceq g$ and $g \preceq f$.

Exercise

- If $f : \mathcal{G} \rightarrow \mathcal{G}'$ is a quasi-isometry then $\mathfrak{G}_{\mathcal{G},x} \asymp \mathfrak{G}_{\mathcal{G}',f(x)}$.
- $\mathfrak{G}_{\mathcal{G},x} \asymp \mathfrak{G}_{\mathcal{G},x'}$ for all $x, x' \in V$.

Consequence= the growth function of a group well defined up to \asymp .

Amenability and growth II

- ① If $G = \mathbb{Z}^k$ then $\mathfrak{G}_G \asymp x^k$.
- ② If $G = F_2$ then $\mathfrak{G}_G(n) \asymp e^n$.
- ③ If G is nilpotent then $\mathfrak{G}_G(n) \asymp n^d$. (Bass' Theorem)

Construct inductively:

$$C^1 G = G, \quad C^{n+1} G = [G, C^n G].$$

The **lower central series** of G is

$$G \geq C^2 G \geq \dots \geq C^n G \geq C^{n+1} G \geq \dots$$

G is (**k -step**) **nilpotent** if there exists k such that $C^{k+1} G = \{1\}$. The minimal such k is the **class** of G .

Examples

- ① An abelian group is nilpotent of class 1.
- ② The group of upper triangular $n \times n$ matrices with 1 on the diagonal is nilpotent of class $n - 1$.

Amenability and growth III

- ① The growth function is **sub-multiplicative**:

$$\mathfrak{G}_S(r+t) \leq \mathfrak{G}_S(r)\mathfrak{G}_S(t).$$

- ② If $|S| = k$ then $\mathfrak{G}_S(r) \leq k^r$.

- ③ the limit

$$\gamma_S = \lim_{n \rightarrow \infty} \mathfrak{G}_S(n)^{\frac{1}{n}},$$

exists, called **growth constant**.

Amenability and growth IV

If $\gamma_S > 1$ then G is said to be of **exponential growth**.

If $\gamma_S = 1$ then G is said to be of **sub-exponential growth**.

A graph \mathcal{G} is of **sub-exponential growth** if for some basepoint $x_0 \in V$

$$\limsup_{n \rightarrow \infty} \frac{\ln \mathfrak{G}_{x_0, X}(n)}{n} = 0.$$

For every other basepoint y_0 , $\mathfrak{G}_{y_0, X}(n) \leq \mathfrak{G}_{x_0, X}(n + \text{dist}(x_0, y_0))$, hence definition independent of the choice of basepoint.

Amenability and growth V

Proposition

A non-empty graph \mathcal{G} of bounded geometry and sub-exponential growth is amenable: for every $v_0 \in V$ there exists a Følner sequence consisting of metric balls with center v_0 .

Amenability and quasi-isometry

Theorem (R. Brooks)

Let M be a complete connected n -dimensional Riemannian manifold and \mathcal{G} a graph, both of bounded geometry. Assume that M is quasi-isometric to \mathcal{G} . Then the Cheeger constant of M is strictly positive if and only if \mathcal{G} is non-amenable.

Riemannian manifold of bounded geometry=uniform upper and lower bounds for the sectional curvature

Cheeger constant for M : infimum over $h > 0$ such that for every open submanifold $\Omega \subset M$ with compact closure and smooth boundary,

$$\text{Area}(\partial\Omega) \geq h \text{Vol}(\Omega).$$

particular case=when M universal cover of a compact Riemannian manifold C and \mathcal{G} Cayley graph of the fundamental group of C .

Theorem (Graph amenability is QI invariant)

Suppose that \mathcal{G} and \mathcal{G}' are quasi-isometric graphs of bounded geometry. Then \mathcal{G} is amenable if and only if \mathcal{G}' is.

Theorem (K. Whyte)

Let $\mathcal{G}_i, i = 1, 2$, be two non-amenable graphs of bounded geometry. Then every quasi-isometry $\mathcal{G}_1 \rightarrow \mathcal{G}_2$ is at bounded distance from a bi-Lipschitz map.

At the core of this course is the discussion of:

Conjecture (von Neuman-Day conjecture)

Is every finitely generated group either amenable or containing a free non-abelian subgroup ?

Theorem (K. Whyte)

Let \mathcal{G} be an infinite graph of bounded geometry. The graph \mathcal{G} is non-amenable if and only if there exists a free action of F_2 on \mathcal{G} by bi-Lipschitz maps which are bounded perturbations of the identity.

Amenability for groups

A **mean** on a set X = a linear functional $m : \ell^\infty(X) \rightarrow \mathbb{C}$ s.t.

(M1) if f takes values in $[0, \infty)$ then $m(f) \geq 0$;

(M2) $m(\mathbf{1}_X) = 1$.

TFAE in a group G

- 1 there exists a mean m on G invariant by left multiplication.
- 2 there exists a finitely additive probability measure μ on $\mathcal{P}(G)$, the set of all subsets of G , invariant by left multiplication.

A group G is **amenable** if any of the above is true.

Left, right or both

Proposition

- (a) *Invariant by left multiplication (left-invariance) can be replaced by right-invariance.*
- (b) *Moreover, both can be replaced by bi-invariance.*

Proof.

- (a) It suffices to define $\mu_r(A) = \mu(A^{-1})$ and $m_r(f) = m(f_1)$, where $f_1(x) = f(x^{-1})$.
- (b) Let μ be a left-invariant f.a.p. measure and μ_r the right-invariant measure in (a). Then for every $A \subseteq X$ define

$$\nu(A) = \int \mu(Ag^{-1}) d\mu_r(g).$$



Metric and group amenability

Theorem

Let G be a finitely-generated group. TFAE:

- 1 G is amenable;
- 2 one (every) Cayley graph of G is amenable.

Corollary

A finitely generated group is either paradoxical or amenable.