Amenable groups, Jacques Tits' Alternative Theorem

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Last lecture

- Equivalent definitions of non-amenable graphs:
  - positive Cheeger constant: $\inf_{F \text{ finite}} \frac{|\partial_V F|}{|F|} > 0$;
  - expansion condition: $|\overline{\mathcal{N}}_C(F)| \geq 2|F|$ for some $C > 0$.
  - $\forall v \in V$, $f^{-1}(v)$ contains exactly two elements, for some $f \in \mathcal{B}(V)$;
  - (Gromov's condition) $\forall v \in V$, $f^{-1}(v)$ contains at least two elements, for some $f \in \mathcal{B}(V)$.
- a graph of bounded geometry and sub-exponential growth is amenable, with Følner sequences composed of balls.
- amenability is a quasi-isometry invariant (e.g. between Cayley graphs of fundamental groups and universal covers of compact Riemannian manifolds).

TFAE in a group G

1. there exists a mean m on G invariant by left multiplication.
2. there exists a finitely additive probability measure μ on $\mathcal{P}(G)$ invariant by left multiplication.

A group G is amenable if any of the above is true.

Remark

The invariance by left multiplication may be replaced by the invariance by right multiplication, or by the invariance by both left and right multiplication.

Metric and group amenability

Theorem

Let G be a finitely-generated group. TFAE:

1. *G is amenable;*
2. *one (every) Cayley graph of G is amenable.*

Corollary

A finitely generated group is either paradoxical or amenable.

(1) $\Rightarrow$ (2) If some $\mathrm{Cayley}(G, S)$ is non-amenable then $\exists f \in \mathcal{B}(G)$ with pre-images having 2 elements.

Modulo the equivalence in the Theorem, corollary proven.

A useful tool

We prove (2) $\Rightarrow$ (1): given a Følner sequence on a Cayley graph, construct μ invariant measure on G.

Goal: a new notion of limit for sequences in compact spaces (and later for sequences of spaces and of actions of groups.)

Definition

An ultrafilter on a set I = a finitely additive probability measure $\omega : \mathcal{P}(I) \to \{0, 1\}$.

Example

$\delta_x(A) = 1$ if x in A, 0 otherwise.
Called principal (or atomic) ultrafilter.

Ultralimit

Definition

Consider $f : I \to Y$ topological space.
$y \in Y$ is the ω–limit of f, $\lim_\omega f(i)$, if $\forall U$ neighborhood of y, $\omega(f^{-1}U) = 1$.

Theorem

Assume Y compact and Hausdorff. Each $f : I \to Y$ admits a unique ω-limit.

If $\omega = \delta_x$ then $\lim_\omega f(i) = f(x)$.

Theorem

An ultrafilter is non-principal (non-atomic) if and only if $\omega(F) = 0$ for every F finite.

Existence of ultrafilters

Why do non-principal ultrafilters exist ?

Equivalent definition:
A filter $\mathcal{F}$ on a set I is a collection of subsets of I s.t.:

(F_1) $\emptyset \notin \mathcal{F}$;

(F_2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(F_3) If $A \in \mathcal{F}$, $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Example: Complementaries of finite sets in I= the Fréchet filter.

Ultrafilter on I = a maximal element in the ordered set of filters on I with respect to the inclusion.

Non-principal ultrafilter= contains the Fréchet filter.
Exists by Zorn's Lemma.

relation to previous definition: ω is the characteristic function of $\mathcal{U} \subset \mathcal{P}(I)$

Back to the proof

Theorem

Let G be a finitely-generated group. TFAE:

1. *G is amenable;*
2. *one (every) Cayley graph of G is amenable.*

(2)$\Rightarrow$ (1):
A Cayley graph $\mathcal{G}$ is amenable: $\exists$ a Følner sequence $(\Omega_n) \subset G$.

- For every $A \subset G$ define

$$\mu_n(A) = \frac{|A \cap \Omega_n|}{|\Omega_n|}\,.$$

- $|\mu_n(A) - \mu_n(Ag)| \leq \frac{2\partial_V(\Omega_n)}{|\Omega_n|}$ when $g \in S$.
- Let ω be a non-principal ultrafilter on $\mathbb{N}$.
  Take $\mu(A) = \omega\text{-lim}\, \mu_n(A)\,.$

Group operations

Proposition

A subgroup of an amenable group is amenable.

Corollary

Any group containing a free non-abelian subgroup is non-amenable.

1. A finite extension of an amenable group is amenable.

2. Let N be a normal subgroup of a group G. The group G is amenable if and only if both N and G/N are amenable.

3. The direct limit G of a directed system $(H_i)_{i \in I}$ of amenable groups H_i, is amenable.

Corollary

A group G is amenable if and only if all finitely generated subgroups of G are amenable.

Corollary

Every solvable group is amenable.

Solvable groups

$G' =$ derived subgroup $[G, G]$ of the group G.
The iterated commutator subgroups $G^{(k)}$ defined inductively by:

$$G^{(0)} = G, G^{(1)} = G', \dots, G^{(k+1)} = \left(G^{(k)}\right)', \dots$$

The derived series of G is

$$G \trianglerighteq G' \trianglerighteq \dots \trianglerighteq G^{(k)} \trianglerighteq G^{(k+1)} \trianglerighteq \dots$$

G is solvable if there exists k such that $G^{(k)} = \{1\}$.
The minimal k such that $G^{(k)} = \{1\}$ is the derived length of G.

Exercise

The group of upper triangular $n \times n$ matrices in $GL(n, K)$, K a field, is solvable.

Solvable groups continued

Exercise

Suppose G is direct limit of $G_i, i \in I$. Assume that there exist $k, m \in \mathbb{N}$ so that for every $i \in I$, the group G_i contains a solvable subgroup H_i of index $\leqslant k$ and derived length $\leqslant m$. Then G contains a subgroup H of index $\leqslant k$ and derived length $\leqslant m$.

Back to

Corollary

Every solvable group is amenable.

Elementary amenable

Definition

The class of elementary amenable groups $\mathcal{E}\mathcal{A}$= the smallest class containing all finite groups, all abelian groups and closed under finite-index extensions, direct limits, subgroups, quotients and extensions

$$1 \to G_1 \to G_2 \to G_3 \to 1,$$

where both G_1, G_3 are elementary amenable.

There exist Grigorchuk groups of intermediate growth not elementary amenable.

An alternative for $\mathcal{EA}$

Theorem (Chou)

A finitely generated elementary amenable group either is virtually nilpotent or it contains a free non-abelian subsemigroup.

A group G is virtually (***) if it has a finite index subgroup with property (***).

We are now able to relate amenable groups to the Banach–Tarski paradox.

Proposition

1. *The group of isometries* $\mathrm{Isom}(\mathbb{R}^n)$ *with* $n = 1, 2$ *is amenable.*
2. *The group of isometries* $\mathrm{Isom}(\mathbb{R}^n)$ *with* $n \geqslant 3$ *is non-amenable.*

Does there exist a purely algebraic definition of amenability for groups?

Conjecture

Does every non-amenable group contain a free non-abelian subgroup?

The question is implicit in von Neumann's initial paper (1929), formulated explicitly by Day in 1957.

Counter-examples:

- Al. Olshanskii (1980);
- S, Adyan (1982): the free Burnside group $B(n, m)$ with $n \geq 2$ and $m \geq 665$, m odd.

Positive answers

Theorem (Jacques Tits 1972)

A subgroup G of $GL(n, F)$, where F is a field of zero characteristic, is either virtually solvable or it contains a free nonabelian subgroup.

Also true for fields of positive characteristic, if G finitely generated.

Theorem (Mostow–Tits)

A discrete amenable subgroup G of a Lie group L with finitely many components, contains a polycyclic group of index at most $\eta(L)$.

Other classes of groups for which the von Neumann-Day conjecture has a positive answer (in fact Tits' theorem is true):

1. subgroups of Gromov hyperbolic groups (Gromov);
2. subgroups of the mapping class group of a surface (Ivanov 1992);
3. subgroups of $Out(F_n)$ (Bestvina-Feighn-Handel 2000,2004,2005);
4. fundamental groups of compact manifolds of nonpositive curvature (Ballmann 1995).

A metric von Neumann-Day

A metric version of the von Neumann-Day conjecture established by Benjamini and Schramm:

- A locally finite graph $\mathcal{G}$ with positive Cheeger constant contains a tree with positive Cheeger constant.
  Uniform bound on the valency is not assumed.

  Cheeger constant is considered with edge–boundary.

- If, moreover, the Cheeger constant of $\mathcal{G}$ is at least an integer $n \geqslant 0$, then $\mathcal{G}$ contains a spanning subgraph, with each connected component is a rooted tree with all vertices of valency n, except the root, of valency $n+1$.

Metric von Neumann-Day continued

- If X is either a graph or a Riemannian manifold with infinite diameter, bounded geometry and positive Cheeger constant (in particular, if X is the Cayley graph of a paradoxical group) then X contains a bi-Lipschitz embedding of the binary rooted tree.

Related to the above, the following is asked:

Open question

(Benjamini-Schramm 1997) Does every Cayley graph of every finitely generated group with exponential growth contain a tree with positive Cheeger constant?

Open case: amenable non-linear groups with exponential growth.

Next, we shall overview briefly quantitative approaches to:

- non-amenability: Tarski numbers for groups;
- amenability: uniform amenability and Følner functions.