Amenable groups, Jacques Tits' Alternative Theorem

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Last lecture

- For a group, the von Neumann definition (with a mean) is equivalent to the geometric amenability for any Cayley graph;
- a group is either amenable or paradoxical (Taski alternative);
- an extension of the functional lim to sequences in compact spaces, using non-principal ultrafilters.
- group operations preserving amenability $\Rightarrow$ solvable groups are amenable.
- definition of the strictly smaller class of elementary amenable groups: minimal class containing all finite and abelian groups, stable by the same list of group operations.

Quantitative non-amenability

One can measure "how paradoxical" a group G is *via* the Tarski number.

In this discussion, groups are not required to be finitely generated.

Recall that a paradoxical decomposition of a group G is a partition

$$G = X_1 \sqcup \ldots \sqcup X_k \sqcup Y_1 \sqcup \ldots \sqcup Y_m$$

for which $\exists\ g_1, \ldots, g_k, h_1, \ldots, h_m$ in G, so that

$$g_1 X_1 \sqcup \ldots \sqcup g_k X_k = G$$

and

$$h_1 Y_1 \sqcup \ldots \sqcup h_m Y_m = G\,.$$

The Tarski number of the decomposition is $k + m$.

The Tarski number $\mathrm{Tar}(G)$ of the group = minimum of the Tarski numbers of paradoxical decompositions.

If G is amenable then set $\mathrm{Tar}(G) = \infty$.

Tarski numbers and group operations

Proposition

1. $\mathrm{Tar}(G) \geq 4$ *for every group* G.
2. *If* $H \leqslant G$ *then* $\mathrm{Tar}(G) \leqslant \mathrm{Tar}(H)$.
3. $\mathrm{Tar}(G) = 4$ *if and only if* G *contains a free non-abelian sub-group.*
4. *Every paradoxical group* G *contains a finitely generated subgroup* H *with* $\mathrm{Tar}(G)$ *generators, such that* $\mathrm{Tar}(G) = \mathrm{Tar}(H)$.
5. *If* N *is a normal subgroup of* G *then* $\mathrm{Tar}(G) \leqslant \mathrm{Tar}(G/N)$.

Co-embeddable groups

Two groups G_1 and G_2 are co-embeddable if there exist injective group homomorphisms $G_1 \to G_2$ and $G_2 \to G_1$.

1. All countable free groups are co-embeddable.
2. Sirvanjan-Adyan: for every odd $m \geqslant 665$, two free Burnside groups $B(n; m)$ and $B(k; m)$ of exponent m, with $n \geqslant 2$ and $k \geqslant 2$, are co-embeddable.

G_1 (non-)amenable iff G_2 (non-)amenable. Moreover $\mathrm{Tar}(G_1) = \mathrm{Tar}(G_2)$.

Consequence: For every odd $m \geq 665$, and $n \geq 2$, the Tarski number of $B(n; m)$ is independent of the number of generators.

Paradoxical decomposition and torsion

Proposition

1. *If G admits a paradoxical decomposition*

$$G = X_1 \sqcup X_2 \sqcup Y_1 \sqcup \ldots \sqcup Y_m,$$

then G contains an element of infinite order.

2. *If G is a torsion group then $\mathrm{Tar}(G) \geq 6$.*

The Tarski numbers help to classify the groups non-amenable and without an F_2 subgroup ("infinite monsters").

Ceccherini, Grigorchuk, de la Harpe: The Tarski number of a free Burnside group $B(n; m)$ with $n \geqslant 2$ and $m \geqslant 665$, m odd, is at most 14.

Tarski numbers, final

We proved that a paradoxical group G contains a finitely generated subgroup H with $\mathrm{Tar}(G)$ generators, such that $\mathrm{Tar}(G) = \mathrm{Tar}(H)$.

Consequence: if G is such that all m generated subgroups are amenable then $\mathrm{Tar}(G) \geq m + 1$.

M. Ershov: certain Golod-Shafarevich groups G

- have an infinite quotient with property (T);
- for every m large enough, G contains finite index subgroups H_m with the property that all their m-generated subgroups are finite.

Consequences:

- the set of Tarski numbers is unbounded;
- Tarski numbers, when large, are not quasi-isometry invariants. Not even commensurability invariants.

Ershov-Golan-Sapir: D. Osin's torsion groups have Tarski number 6.

Questions

Question

How does the Tarski number of a free Burnside group $B(n; m)$ depend on the exponent m? What are its possible values?

Question

Is the Tarski number of groups a quasi-isometry invariant, when it takes small values?

For $\mathrm{Tar}(G) = 4$ this question is equivalent to a well-known open problem.

A group G is small if it contains no free nonabelian subgroups. Thus, G is small iff $\mathrm{Tar}(G) > 4$.

Question

Is smallness invariant under quasi-isometries of finitely generated groups?

Uniform amenability

Let G be a discrete group. TFAE:

1. G is amenable;
2. (the Følner Property) for every finite subset K of G and every $\epsilon \in (0,1)$ there exists a finite non-empty subset $F \subset G$ satisfying:

$$|KF \triangle F| < \epsilon |F|.$$

uniform Følner Property: $|F|$ has a bound depending only on ϵ and $|K|$:

$\exists\, \phi : (0,1) \times \mathbb{N} \to \mathbb{N}$ such that

$$|F| \leqslant \phi(\epsilon, |K|)\,.$$

Uniform amenability II

Theorem (G. Keller)

A group G has the uniform Følner Property if and only if for some (for every) non-principal ultrafilter ω, the ultrapower G^ω has the Følner Property.

Consider

- $\omega : \mathcal{P}(I) \to \{0,1\}$ non-principal ultrafilter.
- a collection of sets $X_i, i \in I$.

The ultraproduct $\prod_{i\in I} X_i/\omega =$ set of equivalence classes of maps $f : I \to \bigcup_{i\in I} X_i$, $f(i) \in X_i$ for every $i \in I$,

with respect to the equivalence relation $f \sim g$ iff $f(i) = g(i)$ for ω–all i.

The equivalence class of a map f denoted by f^ω.

For a map given by indexed values $(x_i)_{i\in I}$, we use the notation $(x_i)^\omega$.

When $X_i = X$ for all $i \in I \Rightarrow$ the ultrapower of X, denoted X^ω.

Any structure on X (group, ring, order, total order) defines the same structure on X^ω.

When $X = \mathbb{K}$ is either $\mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$, the ultrapower $\mathbb{K}^\omega$ is called nonstandard extension of $\mathbb{K}$;

the elements in $\mathbb{K}^\omega \setminus \mathbb{K}$ are called nonstandard elements.

X can be embedded into X^ω by $x \mapsto (x)^\omega$.

We denote the image of each element $x \in X$ by $\widehat{x}$.

We denote the image of $A \subseteq X$ by $\widehat{A}$.

Internal subsets

Internal subset of an ultrapower X^ω $= W^\omega \subset X^\omega$ s.t. $\forall\, i \in I$ there is a subset $W_i \subset X$ such that

$$f^\omega \in W^\omega \iff f(i) \in W_i \omega - -a.s..$$

Proposition

1. *If an internal subset A^ω is defined by a family of subsets of bounded cardinality $A_i = \{a_i^1, \dots, a_i^k\}$ then $A^\omega = \{a_\omega^1, \dots, a_\omega^k\}$, where $a_\omega^j = \left(a_i^j\right)^\omega$.*
2. *In particular, if an internal subset A^ω is defined by a constant family of finite subsets $A_i = A \subseteq X$ then $A^\omega = \widehat{A}$.*
3. *Every finite subset in X^ω is internal.*

Keller's Theorem 1

We now prove

Theorem (G. Keller)

A group G has the uniform Følner Property if and only if for some (for every) non-principal ultrafilter ω, the ultrapower G^ω has the Følner Property.

Uniform Følner property and laws

G. Keller: A group with the uniform Følner property satisfies a law.

An identity (or law) is a non-trivial reduced word $w = w(x_1, \ldots, x_n)$ in n letters $x_1, \ldots, x_n$ and their inverses.

G satisfies the identity (law) $w(x_1, \ldots, x_n) = 1$ if the equality is satisfied in G whenever $x_1, \ldots, x_n$ are replaced by arbitrary elements in G.

1. Abelian groups. Here the law is

$$w(x_1, x_2) = x_1 x_2 x_1^{-1} x_2^{-1} .$$

2. Solvable groups.
3. Free Burnside groups.

Laws in groups

Proposition

A group G satisfies a law if and only if for some (every) non-principal ultrafilter ω on $\mathbb{N}$, the ultrapower G^ω does not contain a free non-abelian subgroup.

Consequence[G. Keller] Every group with the uniform Følner property satisfies a law.

Question

Is every amenable group satisfying a law uniformly amenable ?

The above is equivalent to von Neumann-Day for ultrapowers of amenable groups.