Amenable groups, Jacques Tits' Alternative Theorem

Cornelia Druțu

Oxford

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Last lecture

- Quantitative non-amenability: Tarski numbers.
  - $\mathrm{Tar}(G) = 4 \Leftrightarrow F_2 \leqslant G$;
  - $\mathrm{Tar}(G) \geq 6$ if G is a torsion group;
  - $\mathrm{Tar}(B(n, m)) \leq 14$ and independent of number of generators n;
  - $\mathrm{Tar}(G) = 6$ for Osin's torsion group G;
  - $\exists\ G$ Golod-Shafarevich paradoxical group such that for every m, $\exists H_m \leqslant G$ finite index, $\mathrm{Tar}(H_m) \geq m$ (M. Ershov).
- Uniform amenability (with Følner condition)implies that G satisfies a law because:
  - uniform amenability $\Leftrightarrow$ amenability of one (every) ultrapower;
  - G satisfies a law $\Leftrightarrow$ one (every) ultrapower does not contain F_2.

Quantitative amenability

Let $\mathcal{G}$ be an amenable graph of bounded geometry.

The Følner function of $\mathcal{G}$: $\mathrm{F}_o^{\mathcal{G}} : (0,\infty) \to \mathbb{N}$, $\mathrm{F}_o^{\mathcal{G}}(x) :=$ minimal cardinality of $F \subseteq V$ finite non-empty s.t.

$$|\partial_V F| \leq \frac{1}{x}|F|\,.$$

Proposition

If two graphs of bounded geometry are quasi-isometric then they are either both non-amenable or both amenable, and their Følner functions are asymptotically equal.

f and g are asymptotically equal ($f \asymp g$) if $f \preceq g$ and $g \preceq f$.

$f \preceq g$ if $f(x) \leq a g(bx)$ for every $x \geq x_0$ for some fixed x_0 and $a, b > 0$.

Følner functions II

Proposition

Let H be a finitely generated subgroup of a finitely generated amenable group G. Then $\mathsf{F}_o^H \preceq \mathsf{F}_o^G$.

How does the Følner function relate to the growth function?

The main ingredient: isoperimetric inequalities.

Isoperimetric inequality in a manifold M = an inequality of the form

$$Vol(\Omega) \leq f(\Omega) g\left(Area \partial \Omega\right) ,$$

where f and g are real-valued functions, g defined on $\mathbb{R}_+$ and Ω arbitrary open submanifold with compact closure and smooth boundary.

Isoperimetric inequality in a graph $\mathcal{G}$ = replace Ω by $F \subseteq V$ finite, volume and area by cardinality.

Varopoulos inequality

Theorem (Varopoulos inequality)

Let $\mathrm{Cayley}(G,S)$ *be a Cayley graph of* G *with respect to* S*, and* $d = |S|$*.*

For every finite $F \subseteq V$*, let* k *be the unique integer such that* $\mathfrak{G}_S(k-1) \leqslant 2|F| < \mathfrak{G}_S(k)$ *. Then*

$$|F| \leqslant 2d\, k\, |\partial_V F| \,, \tag{1}$$

Consequences:

1. If $\mathfrak{G}_G \asymp x^n$ then

$$|F| \leqslant K\, |\partial_V F|^{\frac{n}{n-1}} \,.$$

2. If $\mathfrak{G}_G \asymp \exp(x)$ then

$$\frac{|F|}{\ln |F|} \leqslant K\, |\partial_V F| \,.$$

Følner function and growth

(A_n) sequence of finite subsets quasi-realizes the Følner function if

- $|A_n| \asymp \mathrm{F}_o^G(n)$
- $|\partial_V(A_n)| \leq \frac{a}{n}|A_n|$, for some $a > 0$ and finite generating set S.

Theorem

Let G be an infinite finitely generated group.

1. $\mathrm{F}_o^G(n) \succeq \mathfrak{G}_G(n)$.
2. *The sequence of balls $B(1, n)$ quasi-realizes the Følner function of G if and only if G is virtually nilpotent.*

Relevant construction: wreath product

- How much can the Følner function and the growth function of a group differ?
- Is there a general upper bound for the Følner functions of a group (like the exponential function for growth) ?

$$\bigoplus_{x\in X} G := \{f : X \to G \mid f(x) \neq 1_G \text{ for finitely many } x \in X\} \,.$$

Define

$$\varphi : H \to \left(\bigoplus_{h\in H} G\right) \,,\ \varphi(h)f(x) = f(h^{-1}x) \,,\ \forall x \in H \,.$$

The wreath product of G with H, denoted by $G \wr H$:= the semi-direct product

$$\left(\bigoplus_{h\in H} G\right) \rtimes_{\varphi} H \,.$$

Følner functions

The wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$ is called the lamplighter group.

Theorem (A. Erschler)

Let G and H be two amenable groups and assume that some representative F of F_o^H has the property that for every $a > 0$ there exists $b > 0$ so that $aF(x) < F(bx)$ for every $x > 0$.
Then the Følner function of $G \wr H$ is asymptotically equal to $[\mathrm{F}_o^H(x)]^{\mathrm{F}_o^G(x)}$.

A. Erschler: for every function $f : \mathbb{N} \to \mathbb{N}$, $\exists\, G$ finitely generated, subgroup of a group of intermediate growth (hence G amenable) s.t. $\mathrm{F}_o^G(n) \geq f(n)$ for n large enough.

Alternative Theorem

Theorem (Jacques Tits 1972)

A subgroup G of $GL(n, F)$, where F is a field of zero characteristic, is either virtually solvable or it contains a free nonabelian subgroup.

Remark

One cannot replace 'virtually solvable' by 'solvable'.
Consider the Heisenberg group $H_3 \leqslant GL(3, \mathbb{R})$ and $A_5 \leqslant GL(5, \mathbb{R})$. The group $G = H_3 \times A_5 \leqslant GL(8, \mathbb{R})$

- *is not solvable; A_5 is simple;*
- *does not contain a free nonabelian subgroup: it has polynomial growth.*

Reduction to G finitely generated

Without loss of generality we may assume G finitely generated in the Alternative Theorem.

Two ingredients are needed:

Proposition

Every countable field F of zero characteristic embeds in $\mathbb{C}$.

Theorem

Let V be a $\mathbb{C}$-vector space of dimension n.
There exist $\nu(n), \delta(n)$ so that every virtually solvable subgroup $G \leqslant GL(V)$ contains a solvable subgroup Λ of index $\leqslant \nu(n)$ and derived length $\leqslant \delta(n)$.