Amenable groups, Jacques Tits' Alternative Theorem

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Last lecture

Quantitative amenability: Følner functions.

- quasi-isometry $\Rightarrow$ asymptotic equality;
- the Varopoulos inequality: growth determines an isoperimetric inequality;
- Consequence: Følner function $\succeq$ growth function;
- the sequence $B_G(1, n)$, $n \in \mathbb{N}$, quasi-realizes the Følner function iff G virtually nilpotent;
- no universal upper bound for Følner functions: $\forall f : \mathbb{N} \to \mathbb{N}$, $\exists\ G$ such that $\mathrm{F}_o^G \geq f$.

Alternative Theorem

Theorem (Jacques Tits 1972)

A subgroup G of $GL(n, F)$, where F is a field of zero characteristic, is either virtually solvable or it contains a free nonabelian subgroup.

Remark

Without loss of generality we may assume that G is finitely generated.

Noetherian rings

References: Onishchik & Vinberg and J.E. Humphreys, *Linear Algebraic Groups*.

Let A be a commutative ring. TFAE

1. every ideal in A is finitely generated;
2. the set of ideals satisfies the ascending chain condition (ACC): every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

stabilizes, i.e., for some integer $N \geqslant 1$, $I_k = I_N$ for every $k \geqslant N$.

A commutative ring A as above is called noetherian.

Examples:

- Every field is noetherian.
- Hilbert's ideal basis theorem: If A is noetherian then $A[X_1, ..., X_n]$ is noetherian.

We fix a field $\mathbb{K}$.

Affine algebraic set= subset Z in $\mathbb{K}^n$ defined as

$$Z = \{(x_1, .., x_n) \in \mathbb{K}^n \, ; \; p_j(x_1, .., x_n) = 0, j \in J\},$$

for a collection of polynomials $p_j \in \mathbb{K}[X_1, ..., X_n]$, $j \in J$.

There is a one-to-one map associating to an algebraic subset in $\mathbb{K}^n$ an ideal in $\mathbb{K}[X_1, ..., X_n]$:

$$Z \mapsto I_Z = \{p \in K[X_1, ..., X_n] \, ; \; p|_Z \equiv 0\} \, .$$

- I_Z is the kernel of the homomorphism $p \mapsto p|_Z$ from $K[X_1, ..., X_n]$ to the ring of functions on Z.
- $\mathbb{K}[X_1, ..., X_n]/I_Z$ may be seen as a ring of functions on Z: the ring of polynomials on Z, denoted $\mathbb{K}[Z]$.

1. Every algebraic set is defined by finitely many equations.
2. The set of algebraic subsets of $\mathbb{K}^n$ satisfies the descending chain condition (DCC): every descending chain of algebraic subsets

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_i \supseteq \cdots$$

stabilizes.

A morphism between affine varieties Y in $\mathbb{K}^n$ and Z in $\mathbb{K}^m$ = a map $\varphi : Y \to Z$, $\varphi = (\varphi_1, \dots, \varphi_m)$, such that φ_i is in $\mathbb{K}[Y]$ for every $i \in \{1, 2, \dots, m\}$.

A morphism is induced by a morphism $\tilde{\varphi} : \mathbb{K}^n \to \mathbb{K}^m$, $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_m)$, with $\tilde{\varphi}_i : \mathbb{K}^n \to \mathbb{K}$ a polynomial function $\forall i$.

An isomorphism = an invertible map $\varphi : Y \to Z$ such that both φ and φ^{-1} are morphisms.

The Zariski topology on $\mathbb{K}^n$ is the topology having as closed sets all the algebraic subsets.

The Zariski closure of a subset $E \subset \mathbb{K}^n$ is

$$\{x \in \mathbb{K}^n \mid p(x) = 0, \forall p \text{ vanishing on } E\}.$$

A topological space such that the closed sets satisfy the DCC (or, equivalently, the open sets satisfy the ACC) is called noetherian.

Example: $\mathbb{K}^n$ with the Zariski topology is noetherian.

Properties:

- Every subspace of a noetherian topological space (with the subspace topology) is noetherian.
- Every noetherian topological space X is compact.

We define a strong version of connectedness, relevant in noetherian spaces.

For a topological space X TFAE

1. every open non-empty subset of X is dense in X;
2. two open non-empty subsets have non-empty intersection;
3. X cannot be written as a finite union of proper closed subsets.

A non-empty topological space as above is called irreducible.

Examples:

- $\mathbb{K}^n$ with the Zariski topology is irreducible.
- An algebraic variety Z is irreducible if and only if $\mathbb{K}[Z]$ does not contain zero divisors.

General properties:

- The image of an irreducible space under a continuous map is irreducible.
- The cartesian product of two irreducible spaces is irreducible.
- A subset is irreducible if and only if its closure is irreducible.
- If Y is irreducible and $Y \subseteq A \subseteq \overline{Y}$ then A is irreducible.
- Every irreducible subset is contained in a maximal irreducible subset.
- The maximal irreducible subsets of X are closed and they cover X.

Theorem

A noetherian topological space X is a union of finitely many distinct maximal irreducible subsets $X_1, X_2, \ldots, X_n$ such that for every i, X_i is not contained in $\bigcup_{j\neq i} X_j$.

Moreover, every maximal irreducible subset in X coincides with one of the subsets $X_1, X_2, \ldots, X_n$. This decomposition of X is unique up to a renumbering of the X_i's.

The subsets X_i defined in the Theorem above are called the irreducible components of X.

An algebraic subgroup of $GL(V)$ is a Zariski-closed subgroup of $GL(V)$.

- The binary operation $G \times G \to G$, $(g, h) \mapsto gh$ is a morphism.
- The inversion map $g \mapsto g^{-1}$ is an automorphism.
- The left-multiplication and right-multiplication maps $g \mapsto ag$ and $g \mapsto ga$, by a fixed element $a \in G$, are automorphisms.

Examples:

1. The subgroup $SL(V)$ of $GL(V)$ is algebraic, defined by $\det(g) = 1$.

2. The group $GL(n, \mathbb{K})$ can be identified to an algebraic subgroup of $SL(n+1, \mathbb{K})$ by mapping every matrix $A \in GL(n, \mathbb{K})$ to the matrix

$$\begin{pmatrix} A & 0 \\ 0 & \frac{1}{\det(A)} \end{pmatrix}.$$

   Consequence: it will not matter if we consider algebraic subgroups of $GL(n, \mathbb{K})$ or of $SL(n, \mathbb{K})$.

3. The group $O(V)$ is an algebraic subgroup, as it is given by the system of equations $M^T M = \mathrm{Id}_V$.

4. The stabilizer $O(q)$ of a quadratic form q on V is algebraic (e.g. $O(n)$ for $q = x^1 + \cdots + x_n^2$; the symplectic group $Sp(2k, \mathbb{K})$)

Proposition

If Γ is a subgroup of $SL(V)$ then its Zariski closure $\bar{\Gamma}$ in $SL(V)$ is also a subgroup.

Irreducibility for algebraic groups:

1. An algebraic subgroup of $GL(n, \mathbb{C})$ is irreducible in the Zariski topology if and only if it is connected in the classical topology.
2. A connected (in classical topology) algebraic subgroup of $GL(n, \mathbb{R})$ is irreducible in the Zariski topology.
3. Only one irreducible component of G contains the identity element. This is called the identity component and is denoted by G_0.
4. The subset G_0 is a normal subgroup of finite index in G whose cosets are the irreducible components of G.

1. An algebraic group G over a field $\mathbb{K}$, contains a radical $\mathrm{Rad}G$, which is the largest irreducible solvable normal algebraic subgroup of G.

   A group with trivial radical is called semisimple.

2. The radical is the same if the group is considered with its real or its complex structure.

3. The commutator subgroup of an irreducible algebraic group is irreducible.
   An irreducible algebraic semisimple group coincides with its commutator subgroup.

4. The image of an algebraic irreducible semisimple group under an algebraic homomorphism is an algebraic irreducible semisimple group.

5. Given an algebraic semisimple group G and a representation $G \hookrightarrow GL(V)$, the space V decomposes into a direct sum of G-invariant subspaces so that the action of G on each of these subspaces is irreducible (i.e. no proper G-invariant subspaces).

At the core of J. Tits' Alternative Theorem is an example of ping-pong on a projective space that we now explain.

In what follows $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{K}^n$ is equipped with the standard inner/hermitian product.

Theorem

Every matrix $M \in GL(n, \mathbb{K})$ admits a Cartan decomposition

$$M = UDV,$$

where U, V are in $O(n)$ (respectively in $U(n)$) and D is a diagonal matrix with positive entries arranged in descending order.

Follows from: given any inner/hermitian product q on $\mathbb{K}^n$, there exists a basis

- orthogonal with respect to q;
- orthonormal with respect to the standard inner product

$$x_1\overline{y}_1 + \ldots + x_n\overline{y}_n.$$

Notation

For $v \in V \setminus \{0\}$, $[v]$ is its projection to $\mathbb{P}(V)$.

For $W \subset V$, $[W]$ is the image of $W \setminus \{0\}$ under the projection $V \to \mathbb{P}(V)$.

For a linear map $g : V \to V$, we retain the notation g for the induced projective map $\mathbb{P}(V) \to \mathbb{P}(V)$.

We define an inner/hermitian product on $\mathbb{K}^n \wedge \mathbb{K}^n$ by declaring the basis

$$e_i \wedge e_j, 1 \leqslant i < j \leqslant n+1$$

to be orthonormal. Then

$$\|v \wedge w\|^2 = \|v\|^2 \|w\|^2 - \langle v, w\rangle \langle w, v\rangle .$$

The chordal metric on $\mathbb{PK}^n$ is defined by

$$\mathrm{dist}([v],[w]) = \frac{\|v \wedge w\|}{\|v\| \cdot \|w\|}.$$

- The group $O(n)$ (respectively $U(n)$) preserves the chordal metric.
- The topology induced by the chordal metric is the quotient topology induced from $V \setminus \{0\}$.
- $\mathbb{PK}^n$ with the chordal metric is compact.
- If H is a hyperplane in $\mathbb{K}^n$, given as $\ker f$, where $f : V \to \mathbb{K}$ is a linear functional, then

$$\mathrm{dist}([v],[H]) = \frac{|f(v)|}{\|v\| \, \|f\|}.$$

Proposition

Consider $M \in GL(n, \mathbb{K})$ and $M = UDV$ its Cartan decomposition, where D is a diagonal matrix with diagonal entries $a_1, \dots, a_n$ such that

$$a_1 \geqslant \dots \geqslant a_n.$$

Then M induces a bi-Lipschitz transformation of $\mathbb{P}\mathbb{K}^n$ with Lipschitz constant $\leqslant \frac{a_1^2}{a_n^2}$, Let $g \in GL(n, \mathbb{K})$ be an ordered basis $\{u_1, \ldots, u_n\}$ of eigenvectors, $gu_i = \lambda_i u_i$, such that

$$\lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_{n-1} > \lambda_n > 0\,.$$

Denote $A(g) = [u_1]$ and $H(g) = [\mathrm{Span}\,\{u_2, \ldots, u_n\}]$.

Then $A(g^{-1}) = [u_n]$ and $H(g^{-1}) = [\mathrm{Span}\,\{u_1, \ldots, u_{n-1}\}]$.

Obviously, $A(g) \in H(g^{-1})$ and $A(g^{-1}) \in H(g)$.

Proposition (projective ping-pong)

Assume that g and h are two elements in $GL(n, \mathbb{K})$ diagonal with respect to bases $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$ respectively.

Assume that $A(g^{\pm 1})$ is not in $H(h) \cup H(h^{-1})$, and $A(h^{\pm 1})$ is not in $H(g) \cup H(g^{-1})$.

There exists N such that g^N and h^N generate a free non-abelian subgroup of $GL(n, \mathbb{K})$.