AMENABLE GROUPS; ALTERNATIVE THEOREM EX. SHEET 1
HT 2015

Answers should be sent by email to drutu@maths.ox.ac.uk, in any legible format, by 18 February at the latest

Exercise 1. Recall the concept of an *invariant mean*, which is responsible for the terminology 'amenable':

(1) A *mean* on a group G is a linear functional $m : \ell^\infty(G) \to \mathbb{C}$ defined on the set $\ell^\infty(G)$ of bounded functions on G, with the following properties:
  (M1) if f takes values in $[0, \infty)$ then $m(f) \geq 0$;
  (M2) $m(\mathbf{1}_G) = 1$, where $\mathbf{1}_G$ is the function on G that is constant equal to 1.

  The action of G on $\ell^\infty(G)$ is defined by $g \cdot f(x) = f(g^{-1} \cdot x)$.
  A mean is called *left-invariant* if $m(g \cdot f) = m(f)$ for every $f \in \ell^\infty(G)$ and $g \in G$.

Use the definition of amenability involving the existence of a mean to prove that

(1) a subgroup of an amenable group is amenable;
(2) a paradoxical group is non-amenable.

Exercise 2. Let G be a finitely generated group and S a fixed finite set generating G. The goal of the exercise is to show that the Følner definition of amenability implies the existence of a mean.

Recall that the *Følner definition of amenability* requires that for every finite subset F of G and every $\epsilon > 0$ there exists Ω finite subset in G such that $\Omega \triangle F\Omega$ has cardinality at most $\epsilon \operatorname{card} \Omega$.

(1) Prove that it suffices that the Følner condition is satisfied for the finite set S and arbitrary $\epsilon > 0$.
(2) Let ω be an ultrafilter on $\mathbb{N}$. For every integer $n \geq 1$, let Ω_n be a set such that $\Omega_n \triangle S\Omega_n$ has cardinality at most $\frac{1}{n} \operatorname{card} \Omega_n$. Prove that the map $m : \ell^\infty \to \mathbb{R}$ defined by
$$m(f) = \lim_\omega \frac{1}{\operatorname{card} \Omega_n} \sum_{g \in \Omega_n} f(g)$$
is a G–left–invariant mean. Above $\lim_\omega$ signifies the ω–limit of the sequence.

Exercise 3. Consider $\mathbb{Z}^2$ endowed with the word metric d defined by $\pm(1, 0), \pm(0, 1)$. Prove that, for every non-principal ultrafilter ω, the asymptotic cone of $(\mathbb{Z}^2, \mathrm{d})$ with respect to the centre of observation $(0, 0)$, the scaling sequence $\lambda_n = n$, and ω is isometric to $\mathbb{R}^2$ endowed with the metric defined by the norm $\| \|_1$.

Exercise 4. Let F_2 be the free group with two generators and d the word metric defined by these generators. Prove that, for every non-principal ultrafilter ω and every scaling sequence (λ_n) diverging to ∞, the asymptotic cone $C_{\omega,\lambda}$ of (F_2, d), with respect to the scaling sequence (λ_n), ω and the sequence of centres of observation constant equal to the identity, is a real tree. Prove that moreover for every point x in the cone $C_{\omega,\lambda}$, $C_{\omega,\lambda} \setminus \{x\}$ has continuously many connected components.