AMENABLE GROUPS; ALTERNATIVE THEOREM EX. SHEET 2
HT 2015

Answers should be sent by email to drutu@maths.ox.ac.uk, in any legible format, by 5 March at the latest

Exercise 1. Prove that a group quasi-isometric to $\mathbb{Z}^2$ is virtually $\mathbb{Z}^2$.

Exercise 2. *The goal of the exercise is to prove that ultralimits of asymptotic cones are asymptotic cones.*

Let ω be a non-principal ultrafilter on $\mathbb{N}$ and $\mu = (\mu_n)_{n\in\mathbb{N}}$ a sequence of non-principal ultrafilters on $\mathbb{N}$.

We define $\omega\mu$ on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ such that for every subset A in $\mathbb{N}\times\mathbb{N}$, $\omega\mu(A)$ is equal to the ω–measure of the set of all $n \in \mathbb{N}$ such that $\mu_n(A \cap (\{n\}\times\mathbb{N})) = 1$.

(1) Prove that $\omega\mu$ is a non-principal ultrafilter over $\mathbb{N}\times\mathbb{N}$.

(2) For every doubly indexed and bounded family of real numbers α_{ij}, $i \in \mathbb{N}, j \in \mathbb{N}$ prove that

$$\omega\mu\text{-}\lim \alpha_{ij} = \omega\text{-lim}\left(\mu_i\text{-lim}\alpha_{ij}\right),$$

where the second limit on the right hand side is taken with respect to $j \in \mathbb{N}$.

(3) Let X be a metric space. Consider double indexed families of points $\boldsymbol{e} = (e_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in X and of positive real numbers $\boldsymbol{\lambda} = (\lambda_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ such that

$$\mu_i\text{-lim}\lambda_{ij} = \infty$$

for every $i \in \mathbb{N}$. Let $\mathrm{Cone}_{\mu_i}(X, (e_{ij}), (\lambda_{ij}))$ be the corresponding asymptotic cone of X.

Prove that the map

$$\omega\mu\text{-lim}\,(x_{ij}) \mapsto \omega\text{-lim}\,(\mu_i\text{-lim}\,(x_{ij}))\,,$$

is an isometry from $\mathrm{Cone}_{\omega\mu}(X, \boldsymbol{e}, \boldsymbol{\lambda})$ onto

$$\omega\text{-lim}\left[\mathrm{Cone}_{\mu_i}(X, (e_{ij}), (\lambda_{ij}))\,, \mu_i\text{-lim}\, e_{ij}\right].$$

Exercise 3. *We prove a converse of a step in the proof of Gromov's Theorem:* if a finitely generated group G (with a fixed word metric on it) is such that all its asymptotic cones are proper then its growth is *almost polynomial*, that is there exist $C > 0$ and $d \in \mathbb{N}$ such that

$$\mathrm{card}\, B_G(1, n) \leq Cn^d\,, \forall n \in \mathbb{N}.$$

(1) Arguing by contradiction, find a sequence of radii R_n such that the cardinalities $\mathrm{card}\, B_G(1, R_n)$ increase particularly fast.

(2) Deduce from the hypothesis applied to a cone $\mathrm{Cone}_\omega(G, 1, (R_n))$, from the natural onto map $G^\omega \mapsto \mathrm{Cone}_\omega(G, 1, (R_n))$ (where G^ω is the ultrapower) and from Los's Theorem that there exists $k \in \mathbb{N}$ and $\lambda \in (0, 1)$ such that ω– almost surely

$$\mathrm{card}\, B_G(1, R_n) \leq k\mathrm{card}\, B_G(1, \lambda R_n)\,.$$

(3) Show that for an appropriate choice of the sequence (R_n) in (1), the inequality in (2) yields a contradiction.