AMENABLE GROUPS; ALTERNATIVE THEOREM EX. SHEET 3
HT 2015

Answers should be sent by email to drutu@maths.ox.ac.uk, in any legible format, by 19 March at the latest

Exercise 1. Let G be a finitely generated subgroup endowed with a word metric d.

Fix $r > 0$ and an arbitrary finite subset F in G. Prove that the displacement function $G \to \mathbb{R}$, $x \mapsto \Delta(F, x, r)$ is 2–Lipschitz.

Exercise 2. Prove that if a finitely generated group G has Kazhdan's property (T) then its abelianization $G^{ab} = G/[G, G]$ is finite.

Exercise 3. Prove that an arbitrary free finitely generated non-abelian group is a-T-menable.

Exercise 4. Let G and H be finitely generated groups, and let $\varphi_n : G \to H$ be a sequence of homomorphisms such that for $n \neq m$, φ_n and φ_m are not conjugate, that is there exists no element $h \in H$ such that

$$\varphi_n = c_h \circ \varphi_m,$$

where $c_h : H \to H, c_h(x) = hxh^{-1}$.

For a fixed word metric d_H in H and a fixed finite set generating G define

$$\delta_n = \inf_{h \in H} \sup_{s \in S} \mathrm{d}_H(\varphi_n(s)h, h) .$$

(1) Prove that δ_n diverges to $+\infty$.
(2) Prove that using the sequence φ_n one can define an isometric action of G on some asymptotic cone of H without global fixed point.
(3) Prove that if G has property (T) and H is Gromov hyperbolic then a sequence of homomorphisms φ_n as described above cannot exist.