AMENABLE GROUPS; ALTERNATIVE THEOREM EX. SHEET 4
HT 2015

Answers should be sent by email to drutu@maths.ox.ac.uk, in any legible format, by 2 April at the latest

Exercise 1. Consider $\mathbb{R}^n$ endowed with the Euclidean norm $\|x\|_2 = \sqrt{\Sigma_{i=1}^n x_i^2}$.

(1) Prove that for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that if $x, y \in \mathbb{R}^n$, $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, $\left\| \frac{1}{2}x + \frac{1}{2}y \right\| \leq 1 - \delta$.

(2) Let G be a finitely generated subgroup of the orthogonal group $O(n)$, and let S be a finite set generating G. Assume that G has Kazhdan's property (T).
Prove that there exists $\lambda \in (0,1)$ such that for every $x \in \mathbb{R}^n$, $\|x\| = 1$,
$$\left\| \frac{1}{|S|} \Sigma_{s \in S} s \cdot x \right\| \leq \lambda\,.$$

Exercise 2. Consider a finitely generated group G and a short exact sequence
$$1 \to N \to G \to H \to 1.$$

(1) Suppose that H and N both have Property (T). Deduce that G has property (T). [*You may assume without proof the fact that a closed subspace of a Hilbert space is itself a Hilbert space*]

(2) Assume that G has property (T). Does this imply that H has property (T) ?

Exercise 3. Let G be a finitely generated group and let H be a finite index subgroup in G.
Prove that

- G has property (T) if and only if H has property (T).
- G is a-T-menable if and only if H is a-T-menable.