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SCHOENBERG: $\forall \psi : X \times X \to \mathbb{R}_+$ CND

$\exists f : X \to \mathcal{H}$ MAP, $\mathcal{H}$ HILBERT:

$$\psi(x,y) = \| f(x) - f(y) \|^2.$$

ψ CND $\Leftrightarrow$ $e^{-t\psi}$, $\forall t>0$ is POS. DEF.

LEFT-INV: $\psi : G \times G \to \mathbb{R}_+$,

$$\psi(gx, gy) = \psi(x,y).$$

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THM. A ASSUME G HAS PROP. (T). THEN EVERY LEFT-INV. CND KERNEL ON G IS BOUNDED.

COR. A.1 IF G HAS (T) THEN EVERY ISOMETRIC ACTION ON $\mathcal{A} \subseteq \mathcal{H}$ HAS BOUDED ORBITS.

PROOF $G \curvearrowright \mathcal{A}$

TAKE $\forall a \in \mathcal{A}$, $\psi_a(g,h) = \|ga - ha\|^2$

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PROOF OF THM. A ASSUME $\exists$
$\psi : G \times G \to \mathbb{R}_+$ CND, UNBOUNDED.
$\forall t > 0$, $\varphi_t = e^{-t\psi}$ is P.D., LEFT-i.
DIXMIER: $\exists \rho_t : G \to \mathcal{U}(\mathcal{H}_t)$,
$\exists v_t$, $\|v_t\| = 1$ S.T.

$$e^{-t\psi(g,h)} = \langle \rho_t(g) v_t, \rho_t(h) v_t \rangle.$$

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$\forall s \in S$, $e^{-t\psi(1,s)} \xrightarrow[t\to 0]{} 1$

i.e. $\sup_{s\in S} \langle \rho_t(s) v_t, v_t \rangle \xrightarrow[t\to 0]{} 1$.

$\iff \sup_{s \in S} \| \rho_t(s) v_t - v_t \| \to 0$.

G HAS (T), FOR t SMALL ENOUGH,

$\rho_t(G)$ HAS FIXED VECTORS.

DENOTE THEIR SP. $\mathcal{H}_t^G$.

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THEN $\|v_t'\| = \delta_t \to 0 \ (t \to 0)$.

$\Rightarrow \|v_t - w_t\| \leq \delta_t \to 0$

$\forall g \in G$, $\|\rho_t(g) v_t - v_t\| \leq$

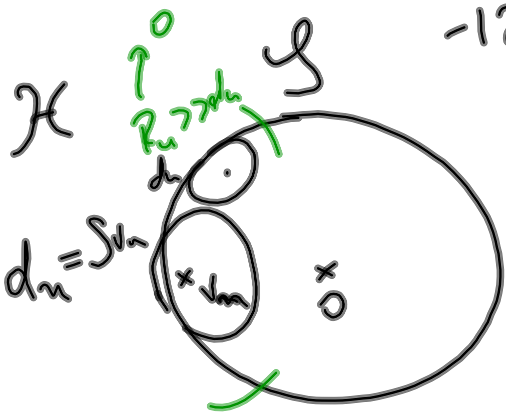
$$\leq 2\delta_t + \underbrace{\|\rho_t(g) w_t - w_t\|}_{0}$$

BUT WE HAD $\exists g_t$ S.T.

$$\|\rho_t(g_t) v_t - v_t\| \geq 1.$$

$\lightning$

THUS ψ MUST BE BOUNDED.



$$\mathcal{S} = \{ v \, ; \, \|v\| = 1 \}$$

$$\text{RESCALE} \left(\mathcal{S}, \frac{1}{d_n} \| \ \| \right)$$

THIS SP. IS ISOM. TO

$$\mathcal{S}_{\frac{1}{d_n}} = \left\{ v \, ; \, \|v\| = \frac{1}{d_n} \right\}$$

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ASSUME, ON THE CONTRARY:

$\forall y \in \mathcal{Y} \cap B(x, 2\delta \,\mathrm{DIAM}(S_x))$,

$\exists z \in \mathcal{Y} \cap B(y, \frac{\delta}{2} \mathrm{DIAM}(S_y))$

S.T. $\mathrm{DIAM}(S_z) < \frac{\mathrm{DIAM}(S_y)}{2}$.

TAKE $x_0 = x = y$.

$\exists z_0 \in \mathcal{Y} \cap B(x, \frac{\delta}{2} \mathrm{DIAM}(S_x))$

S.T. $\mathrm{DIAM}(S_{z_0}) < \frac{\mathrm{DIAM}(S_x)}{2}$.

TAKE $x_1 = z_0 = y$;

$\exists z_1 \in \mathcal{Y} \cap B(x_1, \frac{\delta}{2} \mathrm{DIAM}(S_{x_1}))$

S.T. $\mathrm{DIAM}(S_{z_1}) < \frac{\mathrm{DIAM}(S_{x_1})}{2}$

$< \frac{\mathrm{DIAM}(S_x)}{2^2}$.

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$$d(x_n, x_{n+1}) < \frac{\delta}{2} \frac{\mathrm{DIAM}(S_x)}{2^n}$$

$\Rightarrow (x_n)$ CAUCHY, CONV. TO $u \in \mathcal{Y}$.

$\mathrm{DIAM}(S x_n) \to 0 \Rightarrow S_u = u$.

THUS OUR STATEMENT (IN BLUE) IS TRUE.

WE APPLY IT TO

$x = v_n$ S.T. $\mathrm{DIAM}(S v_n) = a_n \to 0$.

$$\delta_n = \frac{1}{\sqrt{a_n}} \to +\infty$$

$$\delta_n a_n = \sqrt{a_n} \to 0.$$

$$\exists\, y_n \in \mathcal{Y} \cap B(v_n, 2\sqrt{a_n})$$

$$\text{S.T. } \forall z_n \in B\left(y_n, \frac{\delta}{2} \mathrm{DIAM}(S y_n)\right)$$

$$\mathrm{DIAM}(S z_n) \geq \frac{\mathrm{DIAM}(S y_n)}{2}$$

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NOW WE TAKE AN ULTRALIMIT

$$\mathcal{S}\quad \lim_{\omega}\left(\mathcal{S}, y_n, \frac{1}{b_n} d_{\mathcal{X}}\right)$$

$$= \lim_{\omega}\left(\mathcal{S}_{\lambda_n}, c_n\right),$$

[NOTATION: $\lambda_n = \frac{1}{b_n} \to \infty \quad (n \to \infty)$]

WHERE $\mathcal{S}_{\lambda_n} = \{ x \,;\, \|x\| = \lambda_n \}$

$$c_n = \lambda_n y_n .$$

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LEMMA $\lim_\omega \left(\mathcal{Y}_{\lambda_n} \cap B(c_n, \delta_n), c_n \right) =$
$= \lim_\omega \left(\mathcal{Y}_{\lambda_n}, c_n \right)$ IS A HILBERT SP.

PROOF ① TAKE x_n S.T. $\lim_\omega x_n$ IS IN $\lim_\omega \left(\mathcal{Y}_{\lambda_n}, c_n \right)$.

THEN $d_X(x_n, c_n) \leq M_x$ ω-a.s.

FOR n LARGE ENOUGH,

$\delta_n \geq M_x \Rightarrow x_n \in \mathcal{Y}_{\lambda_n} \cap B(c_n, \delta_n)$.

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THEN $\lim_\omega x_n = \lim_\omega x_n' \in \lim_\omega J_n$

THUS $\lim(\mathcal{J}_{\lambda_n}, c_n) = \lim_\omega (J_n, c_n)$,

J_n HAS AN INDUCED HILBERT STR.

REM. ULTRALIMITS OF HILBERT SP. ARE HILBERT SP.

THUS $G \curvearrowright \lim_\omega J_n = J_\omega$

ISOM., WITHOUT FIXED PT.

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AT THE OTHER END:

THEOREM EVERY AMENABLE GP. IS a-T-MENABLE (I.E. IT HAS A PROPER ACTION ON SOME HILBERT SPACE).

PROOF (ACKEMAN-WALTER).

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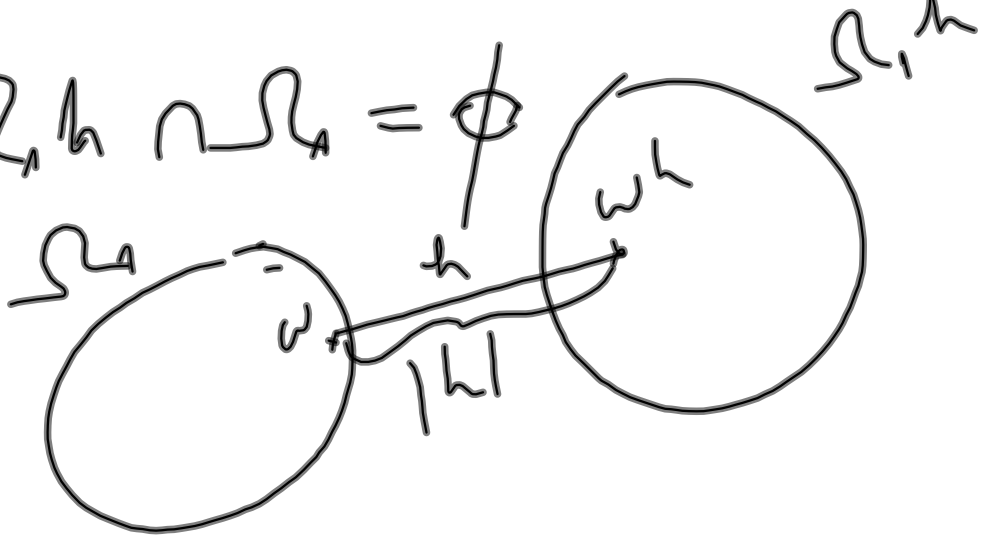
TAKE $B_1 = B(e,1)$. $\varepsilon = \frac{1}{4}$:

$\exists\ \Omega_1$ S.T. $|\Omega_1 B_1 \Delta \Omega_1| \le \frac{1}{4}|\Omega_1|$

Ω_1 FINITE $\longrightarrow \delta_1 = \mathrm{DIAM}(\Omega_1)$

THEN $\forall\, h \notin B(1, \delta_1 + 1)$,

$\Omega_1 h \cap \Omega_1 = \emptyset$



=28=

CONSIDER $\mathcal{H} = \ell^2(G)$.

TAKE $G \curvearrowright \mathcal{H}$: $g \cdot f = f \circ R_g$

$$f_n = \frac{1}{\sqrt{|\Omega_n|}} \chi_{\Omega_n} \in \mathcal{H}, \ \|f_n\| = 1.$$

WE DEFINE A PROPER, LEFT-INV. CND KERNEL ON G:

$$\psi(g,h) = \sum_{n=1}^{\infty} 2^n \|g f_n - h f_n\|^2$$

INV. CND KERNEL.

$$=30=$$

WHEN $g \notin B_k$, $\Omega_k g \cap \Omega_k = \phi$

$$\|g f_k - f_k\| = \frac{1}{|\Omega_k|}\|g \chi_{\Omega_k} - \chi_{\Omega_k}\|^2 =$$

$$= \frac{1}{|\Omega_k|} 2|\Omega_k| = 2$$

$\psi(g,1) \geq$ ITS K-TH TERM $2^k \cdot 2$.

$\boxed{\text{QED}}$