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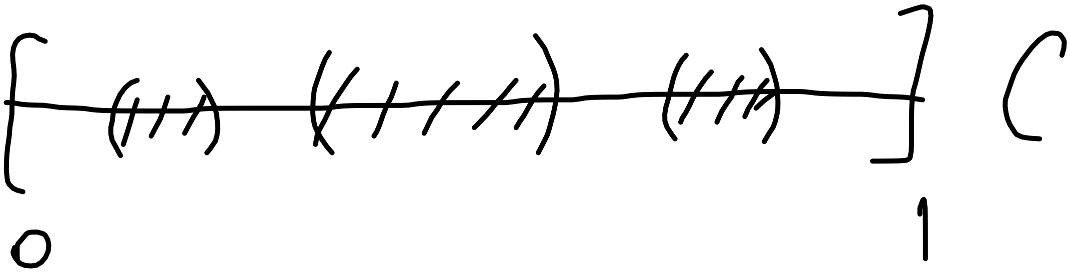
# HAUSDORFF DIM.

## CANTOR SET:




. $\mathrm{INT}(C) = \emptyset \Rightarrow$

$\Rightarrow \quad \mathrm{DIM}\ C < 1$

. UNCOUNTABLE

$\Rightarrow \mathrm{DIM}\ C > 0$

$$3C = C \sqcup C$$

$$3^{\mathrm{DIM}} = 2 \Leftrightarrow \mathrm{DIM} = \frac{\ln 2}{\ln 3}$$

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$\mathcal{H}_d$ IS A COUNTABLE MEASURE WITH POSSIBLE VALUE $+\infty$.

CALLED THE $d$-DIM. HAUSDORFF MEASURE.

FOR $X = \mathbb{R}^n$, $\mathcal{H}_n$ is AN EXTENSION OF LEB.

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PROPOSITION

$$\dim_H\left(\bigcup_{n\in\mathbb{N}} S_n\right) = \sup_{n\in\mathbb{N}} \dim_H S_n$$

COR. $(X,d)$ PROPER AND HOMOGENEOUS ($\mathrm{Isom}(X)$ ACTS TRANSITIVELY).

$$\dim_H X = \dim_H B(x,r),$$

$\forall x, \forall r>0$.

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LEMMA CONSIDER $(X, d)$ s.t. $\exists\, \delta > 0$ WITH THE PROPERTY:

$\forall \varepsilon > 0$, $\forall F \subseteq X$, $F$ $\varepsilon$-SEPARATED SUBSET, $\# F \leq \left(\frac{1}{\varepsilon}\right)^{\delta}$.

THEN $\dim_H X \leq \delta$.

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LAST COURSE : PROVED THAT

$\exists\ (\eta_n)$ DIV. TO $+\infty$ S.T. IN

$$X = \mathrm{CONE}_\omega(G, 1, (\eta_n)),$$

THE BALL $B(1_\omega, \frac{1}{4})$ :

$\forall i \in \mathbb{N},\ i \geqslant 4$, EVERY $\frac{1}{i}$-SEP. SET

HAS CARD. $\leq i^{d+\varepsilon}$.

HERE : $\varepsilon > 0$ ARBITRARY

$d$ S.T. $\mathrm{GROWTH}(G) \leq x^d$.

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ALL HYP. IN MONTGOMERY-ZIPPIN THM. ARE CHECKED $\Rightarrow$ $\mathrm{ISOM}(X) = L$ IS A LIE GP. WITH FIN. MANY CONN. COMP. AND S.T. $\exists\ \varphi : \underbrace{L_e}_{\text{CONN. COMP. OF ID, } e} \to GL(n, \mathbb{C})$

WITH $\mathrm{KER}\,\varphi \leq Z(L)$.

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G ACTS ON ITSELF BY ISOM.

$\Rightarrow$ ACTS BY ISOM. ON $X$:

$$\lim_\omega (x_n) \longmapsto \lim_\omega (g x_n).$$

THUS $\exists\ \rho : G \to \mathrm{ISOM}(X) = L$.

BY EVENTUALLY REPLACING G WITH A FINITE INDEX SUBGP:

WE MAY ASSUME

$$\rho : G \to L_e.$$

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$$1 \to \underbrace{\ker \varphi \cap \hat{G}}_{\text{ABELIAN}} \to \hat{G} \to \underbrace{\varphi(\hat{G})}_{\text{VIRT. NILP.}} \to 1$$

$\Rightarrow$ $\hat{G}$ VIRTUALLY SOLVABLE

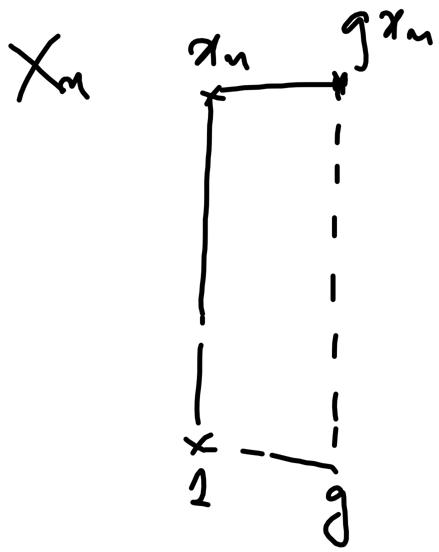
$\Rightarrow$ (MILNOR-WOLF) $\hat{G}$ VIRT. NILPOTENT.

TWO THINGS YOU NEED TO KNOW ABOUT NILP. GROUPS:

(N1) EVERY F.G. NILPOTENT GP. CONTAINS A F. INDEX SUBGP. TORSION-FREE (ALL ELEM. OF $\infty$ ORDER)

(N2) A TORSION-FREE $\infty$ NILPOTENT GP. HAS $\infty$ ABELIANIZATION

$$G^{ab} := G/[G,G].$$

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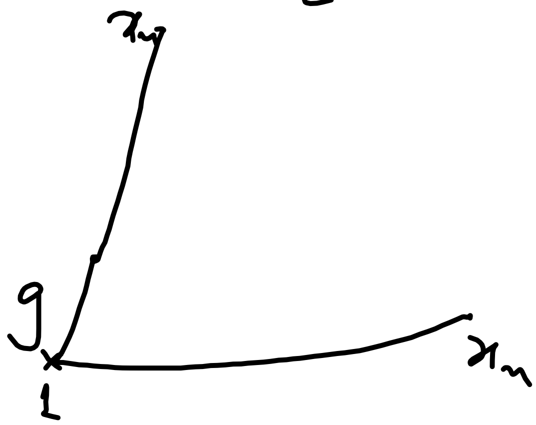
FOR $G = F_2$




**CASE 1** $\hat{G}$ IS INFINITE.

$\hat{G}^{ab}$ $\infty$, $= \mathbb{Z}^m \times$ FINITE.

WE OBTAIN A SURJ. MAP

$$\pi : G \to \mathbb{Z}.$$

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PROOF $G = \langle S \rangle$.

TAKE $\gamma \in \pi^{-1}(1)$.

$\forall s \in S$, $\exists\, m_s \in \mathbb{Z}$ s.t. $\pi(s\gamma^{m_s}) = 0$.

DENOTE $g_s = s\gamma^{m_s} \in K$.

$\{g_s ; s \in S\} \cup \{\gamma\}$ GEN. $G$.

(A1) $\{\gamma^m g_s \gamma^{-m} ; s \in S, \forall m \in \mathbb{Z}\}$

GENERATE $K$.

$$\overset{=20=}{\text{(A2) FIX } s \in S.}$$

DEFINE A SEQ. OF MAPS:

$$\prod_{i=0}^{m} \mathbb{Z}_2 \longrightarrow K$$

$$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

$$(a_i) \longmapsto \gamma g_s^{a_0} \gamma g_s^{a_1} \cdots \gamma g_s^{a_m}$$

IF ALL INJECTIVE:

IN $B(1, n+1)$ WE FOUND

$2^{n+1}$. $\#$

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$$g_{s,n+1}^{b_n - a_n} = \text{prod. of } g_{s,1}, \dots, g_{s,n},$$

$b_n, a_n \in \{0, 1\}$, DISTINCT $\Rightarrow$

$b_n - a_n = \pm 1.$

$$g_{s,n+1} = \gamma^{n+1} g_s \gamma^{-(n+1)}$$

THUS K GEN. BY FINITELY MANY $g_{s,n}$.

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THUS WE CAN ARGUE BY IND. ON $d$
FOR GROMOV'S THM.:

$d=0 \Rightarrow G$ FINITE.

ASSUME GROWTH $\leq x^{d-1} \Rightarrow$ VIRT. NILP.

TAKE $G$, GROWTH $\leq x^{d}$

IF CASE 1, $\hat{G}$ $\infty$, THEN $\exists\, \pi : G \twoheadrightarrow \mathbb{Z}$

APPLY PROP: $1 \to \underbrace{K}_{\text{VIRT. NILP.}} \to G \to \mathbb{Z} \to 1$

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$\forall F \subseteq G$ FINITE

$$\Delta(F; x, r) = \sup_{g \in F} \Delta(g; x, r).$$

IF $x = 1$, WE OMIT IT.

IMMEDIATE PROP.

(1) $\Delta(F; x, r) = \Delta(x^{-1} F x, r)$.

(2) $\forall r > 0$ FIXED, $\forall F$ FIXED

$x \longmapsto \Delta(F; x, r)$ 2-LIPSCHITZ.

(3) $r \longmapsto \Delta(S, r)$ BOUNDED

$\Rightarrow$ G VIRTUALLY ABELIAN.

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ASSUME THEN THAT $\Delta(S,r) \to \infty$ $(r \to \infty)$.

HYPOTHESIS IM $\rho = id$ :

$$\frac{\Delta(S, \eta_n)}{\eta_n} \longrightarrow 0$$

FIX SOME $n$.

$\Delta(S, r)$ UNBOUNDED

$\Rightarrow \exists\, y_n \in G, \exists\, s \in S$

S.T. $d(y_n, s y_n) \geq 2 \varepsilon \eta_n$

FOR ANY $\varepsilon > 0$ FIXED.

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THUS WE FOUND $v_n \in G$ S.T.

$$\lim_\omega \frac{\Delta(v_n^{-1} S v_n ; \eta_n)}{\eta_n} = \varepsilon > 0.$$

THUS $\forall \varepsilon$, WE FOUND $(v_n)$.

WE CAN TWIST THE ACTION OF $G$ ON $X$, THE CONE:

$$g \cdot \left(\lim_\omega x_n\right) =$$

$$= \lim_\omega \left( L_{v_n^{-1} g v_n} x_n \right)$$

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BY FIRST PROPERTY FOR LIE GPS.

$\exists q \in \mathbb{N}$ S.T. $\forall \varepsilon$, $\exists A_\varepsilon \leq \hat{G}_\varepsilon$ ABELIAN,

OF INDEX $\leq q$.

TAKE $G_\varepsilon = \rho_\varepsilon^{-1}(A_\varepsilon)$ HAS INDEX

$\leq q$ IN $G$.

$$\tilde{G} = \bigcap_{\varepsilon > 0} G_\varepsilon$$ FINITE INDEX IN $G$.

THE SECOND PROP. OF LIE GPS $\Rightarrow$

FOR $\varepsilon > 0$ SMALL ENOUGH, $U_\varepsilon$ DOES NOT CONTAIN NON-TRIVIAL FIN. ABEL. SUBGROUPS OF CARD. $\in M$.

ON THE OTHER HAND: THE CONSTR. OF $\tilde{G}$ IS INDEP. OF $\rho_\varepsilon$:

· WE COULD HAVE TAKEN $\tilde{S}$ GEN. FOR $\tilde{G}$.

ARGUED THAT $\Delta(\tilde{S}; v_n, \eta_n) = \varepsilon \eta_n$.

THEN HERE:

$$\Delta(\tilde{S}; 1_\omega, 1) = \varepsilon$$

THUS $\rho_\varepsilon(\tilde{G}) \neq \{id\}$.