Amenable Groups; Alternative Theorem

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The course is centred around two conjectures that attempt to draw a demarcation line between

- “abelian-like” groups and
- “free-like” groups.

These are

- the Von Neumann-Day conjecture;
- the Milnor conjecture.
In the beginning of the XX-th century measure theory emerged (Lebesgue 1901, 1902). Among the questions asked in those days was the following.

**Question**

*Does there exist a measure on $\mathbb{R}^n$ that is*

- finitely additive;
- invariant by isometries;
- defined on every subset?

The Banach–Tarski Paradox answered this question.
Banach-Tarski Paradox

First some terminology:

- **Congruent subsets** in a metric space = subsets $A, B$ such that $\Phi(A) = B$ for some isometry $\Phi$.

- **Piecewise congruent subsets** in a metric space = subsets $A, B$ such that
  \[
  A = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n, \quad B = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n
  \]
  and $A_i, B_i$ congruent for every $i$. 

Theorem (Banach-Tarski)

Every two subsets $A, B$ in $\mathbb{R}^n$, $n \geq 3$, $A, B$ with non-empty interiors are piecewise congruent.

Proof uses the Axiom of Choice.

Examples:

- $A$ and $\lambda A$, for $\lambda > 0$;
- $B(0, 1)$ and $B(0, 1) \sqcup B(c, 1)$. 

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In view of the Banach-Tarski Paradox, several conclusions are possible:

**Conclusion 1**: The answer to the question on measures on $\mathbb{R}^n$ is negative. In particular there exist subsets in $\mathbb{R}^n$ “without volume”.

**Conclusion 2 (the opposite)**: Hard to believe that sets “without volume” exist, therefore the Axiom of Choice should not be admitted.

Note that

- Banach-Tarski is not provable nor disprovable only with Zermelo-Frankel;
- The Axiom of Choice can be replaced with the Hahn-Banach Extension Theorem (a weaker hypothesis).
Amenable groups

**Conclusion 3:** Why the hypothesis “dimension at least 3” in Banach-Tarski?

John von Neumann (1929) defined amenable groups to explain this.

Let $G$ be locally compact, second countable.

$G$ amenable = there exists a functional $m : \ell^\infty(G) \to \mathbb{R}$ (mean) such that

- $m$ has norm 1;
- if $f \geq 0$ then $m(f) \geq 0$ (m positive definite);
- $m$ invariant by $G$-left translations.
Amenable groups II

Equivalently, for every compact $K \subset G$ and every $\epsilon > 0$ there exists $\Omega \subseteq G$ Haar measurable, of positive measure $\nu(\Omega) > 0$ such that

$$\nu(K\Omega \triangle \Omega) \leq \epsilon \nu(\Omega).$$

Remark

In all the above, left multiplication can be replaced by right multiplication.

Examples

- $G$ finite.
- $G = \mathbb{R}^n$ or $\mathbb{Z}^n$.
  For $G = \mathbb{Z}$, for every $K$ finite and $\epsilon > 0$, $\Omega = [-n, n] \cap \mathbb{Z}$ with $n$ large enough works.
Proposition (immediate properties)

- A subgroup of an amenable group is amenable.
- Given a short exact sequence
  \[ 1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1 \]

  \( G \) amenable iff \( N \) and \( Q \) amenable.
- A direct limit of amenable groups is amenable.

Exercise: Prove the above.
“Abelian-like” groups

Construction by induction: \( C^1(G) := G \), \( C^2(G) = [G, C^1(G)] \), \ldots, \( C^{i+1}(G) = [G, C^i(G)] \).

Thus, a decreasing series of characteristic subgroups

\[
C^1(G) \supseteq C^2(G) \supseteq \cdots \supseteq C^i(G) \supseteq \cdots
\]

called lower central series.

\( G \) nilpotent if \( \exists k \) s.t. \( C^k(G) = \{1\} \).

\( G' = [G, G] \) = the derived group, \( G^{(i+1)} = (G^{(i)})' \).

Another decreasing series of characteristic subgroups

\[
G \supseteq G' \supseteq \cdots \supseteq G^{(i)} \supseteq \cdots
\]

called derived series.

\( G \) solvable if \( \exists k \) s.t. \( G^{(k)} = \{1\} \).
Solvable, paradoxical

**Corollary**

*Every solvable group is amenable.*

Prove the Corollary using Proposition 4.

**G is paradoxical if**

- ∃ a decomposition $G = X_1 \sqcup \cdots \sqcup X_k \sqcup Y_1 \sqcup \cdots \sqcup Y_m$;
- ∃ $g_1, \ldots, g_k$ and $h_1, \ldots, h_m$ in $G$ s.t.
  $$G = g_1X_1 \sqcup \cdots \sqcup g_kX_k = h_1Y_1 \sqcup \cdots \sqcup h_mY_m.$$

**Theorem**

*G is non-amenable if and only if G is paradoxical.*

**Exercise:** Prove that $G$ paradoxical $\Rightarrow$ $G$ non-amenable.
The free group

Free group of rank 2, \( F_2 = \langle a, b \rangle \) = set of words in the alphabet \( \{ a, b, a^{-1}, b^{-1} \} \), reduced (i.e. no \( xx^{-1} \)). Binary operation is concatenation and reduction.

For \( x \in \{ a, b, a^{-1}, b^{-1} \} \), \( W_x \) = set of words beginning in \( x \).

Define \( W'_b = W_b \setminus \{ b^n \mid n \geq 1 \} \), \( W'_{b^{-1}} = W_{b^{-1}} \cup \{ b^n \mid n \geq 0 \} \).

\[
F_2 = W_a \sqcup W_{a^{-1}} \sqcup W_b \sqcup W'_{b^{-1}} = W_a \sqcup aW_{a^{-1}} = W'_b \sqcup bW'_{b^{-1}}.
\]

Corollary

If \( F_2 \leq G \) then \( G \) is non-amenable.
Back to the question on dimension for Banach–Tarski:

- The group of isometries of $\mathbb{R}^n$ is $O(n) \rtimes \mathbb{R}^n$.
- For $n = 2$, $O(2)$ is virtually abelian.
- For $n \geq 3$, $O(n)$ contains $F_2$.

$G$ is virtually $(*) = \exists H \leq G$ of finite index, $H$ with property $(*)$.

Question

*Can one find an algebraic definition for amenability/non-amenability?*

Conjecture of von Neumann-Day: Every finitely generated group is either amenable or it contains (a copy of) $F_2$. 
Theorem (Jacques Tits 1972)

Let $L$ be a Lie group with finitely many connected components. If $G \leq L$, $G$ finitely generated then either $F_2 \leq G$ or $G$ is virtually solvable.

Remark

“Virtually solvable” cannot be replaced with “solvable”: take $G = F \times S$, where $F$ is finite in $\text{GL}(n, \mathbb{R})$, $S$ solvable in $\text{GL}(m, \mathbb{R})$, embed $G$ in $\text{GL}(n + m, \mathbb{R})$ using diagonal blocks.

Exercise: Prove that virtually solvable groups are also amenable.
von Neumann-Day

Counter-examples to general von Neumann-Day:

- Olshanskii (1980), using small cancellation, a technique to construct “infinite monsters”.
- Adyan: Burnside groups are counter-examples:

  \[ B(n, m) = \langle x_1, \ldots, x_n \mid w^m = 1 \rangle \text{ for } n \geq 2, m \geq 665, m \text{ odd}. \]

- Olshanskii, Sapir 2003: finitely presented counter-examples.
- Monod 2012: The group of piecewise projective homeomorphisms of \( \mathbb{R} \) is non-amenable, does not contain \( F_2 \).
Milnor's Conjecture

Let $M$ be a Riemannian manifold. Fix $p \in M$. Consider the function $r \mapsto \text{Vol}(B(p, r))$.

**Questions:**

- Dependence on $p$?
- If $N$ is a compact manifold and $M = \tilde{N}$, can part of the behaviour of the function be detected by looking at $\pi_1(N)$?

Let $f, g : A \to \mathbb{R}, A \subseteq \mathbb{R}$.

- $f \precsim g \iff \exists a, b, c, d, e$ positive s.t. $f(x) \leq ag(bx + c) + dx + e$.
- $f \asymp g \iff f \precsim g, g \precsim f$.

$f, g$ as above are called **asymptotically equal**.
Milnor’s Conjecture II

- Dependence on $p$ is up to $\approx$.

- For $G$ finitely generated define $r \mapsto \#B(1, r)$, with $B(1, r)$ ball centred in 1 for some metric on $G$:

  fix $S$ finite set generating $G$, $1 \notin S, S^{-1} = S$.

  **word metric:** $\text{dist}_S(g, h) =$ shortest word in $S$ representing $h^{-1}g$.

  Growth function on $\tilde{N} \approx$ growth function of $G = \pi_1(N)$ (Milnor; Efremovich- Schwartz).
Examples of growth functions

Examples:

- $\mathbb{Z}^d$ has growth function $\approx x^d$;

- $G$ nilpotent,

$$C^1(G) \supset C^2(G) \supset \cdots \supset C^k(G) = \{1\}.$$ 

Then $C^i(G)/C^{i+1}(G) \cong \mathbb{Z}^{m_i} \times \text{Finite group}$.

Growth of $G \approx x^{\sum_i m_i}$ (Bass-Guivarch).

- the free group $F_2$ has growth $\approx e^x$. 

C. Druțu (Oxford) Amenable Groups; Alternative Theorem TCC 2015, Lecture 1 18 / 25
Theorem (Milnor-Wolf)

If $G$ is finitely generated and solvable then $G$ is either virtually nilpotent or of exponential growth.

Exercise: Prove that for $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$, where $M$ 2 × 2 matrix with integer entries

- either $M$ has an eigenvalue of absolute value $\neq 1$, and growth $\approx e^x$;
- or both eigenvalues have absolute value 1 and $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$ virtually nilpotent.

Milnor’s Conjecture: Every f.g. group has either polynomial or exponential growth.
Proposition

If $G$ has sub-exponential growth then $G$ amenable.

Sub-exponential growth $= \lim_{n \to \infty} n \sqrt[1]{\#B(1, n)} = 1$.

Exercise: prove Proposition.

Hint: try second definition of amenability, with $\Omega = B(1, n)$. 

Milnor’s conjecture

Theorem (M. Gromov’s Polynomial Growth Theorem)

*If a f.g. group has polynomial growth then it is virtually nilpotent.*

Milnor’s conjecture false in general; counter-examples of Grigorchuk, of
groups with growth function sub-exponential and $\gtrsim e^{\sqrt{n}}$. 
Gromov’s theorem

Strategy of proof for Gromov’s theorem:
Let $G$ f.g., growth $\lesssim x^d$.

- Embed $G$ in a Lie group with finitely many connected components.
- Use Alternative Theorem: $G$ either solvable or $F_2 \leq G$. The latter impossible, as growth $\lesssim x^d$.
- Apply Milnor-Wolf Theorem.
Main Tool for Gromov’s Thm.

Tool used to embed $G$ in a Lie Groups:

**Theorem (Montgomery-Zippin)**

*Input*: $X$ metric space that is
- complete;
- connected and locally connected;
- proper (i.e. compact balls);
- of finite Hausdorff dimension.

*Output*: If $H = $ the group of isometries of $X$ acts transitively then
- $H$ has finitely many conn. comp.;
- there exists a homomorphism $\varphi : H_e \to GL(n, \mathbb{C})$ with kernel in the centre $Z(H_e)$. 
**First part of the course**

**Goal:** Make $G$ act on a space $X$.

$G$ already acts on

- $(G$, word metric $)$; but this space is not connected;
- every Cayley graph $\text{Cay}(G, S)$ with the simplicial metric; but for these the group of isometries does not act trasitively.

**Example:** for $G = \mathbb{Z}^2$ and $S = \{ \pm(1, 0), \pm(0, 1) \}$, the Cayley graph $\text{Cay}(G, S)$ is the planar grid with mesh 1.

If we rescale its metric by $\frac{1}{\lambda} \Rightarrow$ planar grid with mesh $\frac{1}{\lambda}$.

The limit of the rescaled grids as $\lambda \to \infty$ is $\mathbb{R}^2$, has all the properties required by Montgomery-Zippin Theorem.

Gromov’s theorem and its proof will occupy the first part of the course.
Second part of the course: Look at actions of $G$ on Hilbert spaces, by affine isometries, i.e. preserving the metric, not necessarily linear.

Masur-Ulam: Every affine isometry of a Hilbert space $H$ is of the form $\nu \mapsto Av + b$, where $A$ unitary transformation, $b \in H$.

Assume $G$ locally compact, second countable.

Kazhdan’s Property (T): Every continuous action by affine isometries on a Hilbert space has a global fixed point.

a-T-menable (Haagerup property): There exists a continuous action by affine isometries on a Hilbert space that is proper: when $g$ leaves every compact, $\|gv\| \to \infty$, for $v$ arbitrary fixed.

Amenable groups are a-T-menable. Groups with Kazhdan’s Property (T) are paradoxical.