TRINITY TERM 2016

Answers should be sent by email to Cornelia.Drutu@maths.ox.ac.uk, in any legible format, by 27 May at the latest.

Exercise 1. Consider $\mathbb{Z}^2$ endowed with the word metric d defined by the set of generators $\{\pm(1,0), \pm(0,1)\}$. Prove that, for every non-principal ultrafilter ω, the asymptotic cone of $(\mathbb{Z}^2, \mathrm{d})$ with respect to the centre of observation $(0,0)$, the scaling sequence $\lambda_n = n$, and ω is isometric to $\mathbb{R}^2$ endowed with the metric defined by the norm $\| \|_1$.

Exercise 2. Let F_2 be the free group with two generators and d the word metric defined by these generators. Prove that, for every non-principal ultrafilter ω and every scaling sequence (λ_n) diverging to ∞, the asymptotic cone $C_{\omega,\lambda}$ of (F_2, d), with respect to the scaling sequence (λ_n), ω and the sequence of centres of observation constant equal to the identity, is a real tree. Prove that moreover for every point x in the cone $C_{\omega,\lambda}$, $C_{\omega,\lambda} \setminus \{x\}$ has continuously many connected components.

Exercise 3. *The goal of the exercise is to prove that ultralimits of asymptotic cones are asymptotic cones.*

Let ω be a non-principal ultrafilter on $\mathbb{N}$ and $\mu = (\mu_n)_{n\in\mathbb{N}}$ a sequence of non-principal ultrafilters on $\mathbb{N}$.

We define $\omega\mu$ on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ such that for every subset A in $\mathbb{N}\times\mathbb{N}$, $\omega\mu(A)$ is equal to the ω–measure of the set of all $n \in \mathbb{N}$ such that $\mu_n(A \cap (\{n\} \times \mathbb{N})) = 1$.

(1) Prove that $\omega\mu$ is a non-principal ultrafilter over $\mathbb{N}\times\mathbb{N}$.

(2) For every doubly indexed and bounded family of real numbers α_{ij}, $i \in \mathbb{N}, j \in \mathbb{N}$, prove that

$$\lim_{\omega\mu} \alpha_{ij} = \lim_{\omega} \left(\lim_{\mu_i} \alpha_{ij} \right),$$

where the second limit on the right hand side is taken with respect to $j \in \mathbb{N}$.

(3) Let X be a metric space. Consider double indexed families of points $\mathbf{e} = (e_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in X, and of positive real numbers $\lambda = (\lambda_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ such that

$$\mu_i\text{-}\lim \lambda_{ij} = \infty$$

for every $i \in \mathbb{N}$. Let $\mathrm{Cone}_{\mu_i}(X, (e_{ij}), (\lambda_{ij}))$ be the corresponding asymptotic cone of X.

Prove that the map

$$\lim_{\omega\mu} (x_{ij}) \mapsto \lim_{\omega} \left(\lim_{\mu_i} (x_{ij}) \right),$$

is an isometry from $\mathrm{Cone}_{\omega\mu}(X, \mathbf{e}, \lambda)$ onto

$$\lim_{\omega} \left[\mathrm{Cone}_{\mu_i}(X, (e_{ij}), (\lambda_{ij})), \lim_{\mu_i} e_{ij} \right].$$