people.maths.ox.ac.uk/drutu/tcc4

- book with M. Kapovich
- Assessment:
  - 3 Ex. Sheets
  - deadline on each
  - email answers to
    drutu@maths.ox.ac.uk

# STANDING ASSUMPTIONS

- CONSIDER TOPOLOGICAL GROUPS $G$ LOCALLY COMPACT, SECOND COUNTABLE
- WE CONSIDER ACTIONS ON METRIC SPACES, BY ISOMETRIES, CONTINUOUS: GIVEN $X$ SPACE:

$\forall x \in X$, THE ORBIT MAP

$$\begin{aligned} G &\to X \\ g &\mapsto gx \end{aligned}$$

IS CONT.

# OVERVIEW OF TOPICS OF COURSE

WILL FOCUS ON 2 FUNDAM. PROP. :

1. KAZHDAN'S PROPERTY (T)
2. HAAGERUP PROPERTY
   (a-(T)-MENABILITY)

1. PROPERTY (T) IS RELEVANT IN MANY SETTINGS:

- IN COMBINATORICS
  (PROVIDES A CONSTRUCTION OF EXPANDERS)

- FOR STRUCTURAL PROP. OF GROUPS (IMPLIES FINITE GENERATION, FINITE ABELIANIZATION, NON-SPLITTING: $\neq A *_C B$, $A *_C$ )
- SMOOTH DYNAMICS:
  - ACTIONS ON $S^1$ (A. NAVAS)
  - LOCAL RIGIDITY (D. FISHER & G. MARGULIS).
- ERGODIC THEORY
- OPERATOR ALGEBRAS, RELEVANT FOR BAUM-CONNES

EXAMPLES OF GROUPS WITH (T):

1. SIMPLE GPS. OF RANK $\geq 2$

   $SL(n,\mathbb{R})$, $n\geq 3$, $SO(n,m)$, $n,m\geq 2$

2. ALL THEIR (ARITHMETIC) LATTICES:

   $SL(n,\mathbb{Z})$, $n\geq 3$, $SO_{\mathbb{Z}}(n,m)$, $n,m\geq 2$.

③ RANDOM GROUPS HAVE (T):

SKETCHY EXPLANATIONS:

CONSIDER GPS. WITH FINITE PRES.:

$$G = \langle S \mid R \rangle$$ I.E. DESCRIBED

BY GIVING • A FINITE GENERATING SET $S$

• A SET OF RELATORS $R$ SATISFIED.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$$

WE ORDER THESE GPS. BY <u>DENSITY</u>:

$$d \in [0,1)$$

$$\#R = (\#S)^d.$$

$d=0$ MEANS THAT $\#R = o(\#S)$.

FOR A GIVEN DENSITY $d$, WE SAY THAT A PROPERTY $P$ IS TRUE A.A.S. (ASYMPTOTICALLY ALMOST SURELY)

IF: - CONSIDER AN INCREASING SEQ. OF FINITE SETS $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \subseteq \ldots$

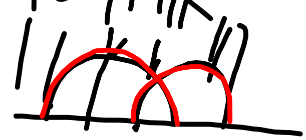
$$\bigcup F_n = \{\text{ALL GPS. WITH DENSITY } d\}$$

- $$\frac{\#\{\text{GPS. IN } F_n \text{ WITH } P\}}{\# F_n} \xrightarrow[n \to \infty]{} 1$$

<u>EX.</u> IF $d > \frac{1}{2}$, A.A.S. A GROUP IS $\{id\}$ OR $\mathbb{Z}/2\mathbb{Z}$.

IF $d < \frac{1}{2}$, A.A.S. $G$ IS <u>GROMOV HYPERB.</u> AND <u>INFINITE</u>.

$\simeq$ A GEOMETRY SIMILAR TO $\mathbb{H}^2_{\mathbb{R}}$



IF $d > \frac{1}{3}$ A.A.S. A GROUP HAS (T).

HAAGERUP PROPERTY

(a-(T)-MENABILITY)

RELEVANT IN:

- BAUM-CONNES CONJ.

MIGSON-KASPAROV: HAAGERUP $\Rightarrow$ B.-C.

- MEASURE GROUP THEORY

EX. OF GROUPS WITH HAAGERUP:

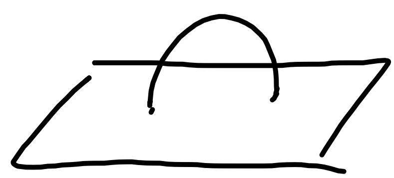
① ALL AMENABLE GPS. (IN SENSE OF VON NEUMANN)

IN PARTICULAR

ABELIAN ⊂ NILPOTENT ⊆ SOLVABLE

② ALL GROUPS OF ISOM. OF $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$

$G = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$



$\forall\, \Gamma \leq G$ DISCRETE, E.G.

$SL(2,\mathbb{Z})$, $SO_{\mathbb{Z}}(n,1)$

③ IF $d < \frac{1}{6}$, A RANDOM GROUP HAS HAAGERUP.

THE APROACHES WE USE:

① STUDY GPS VIA ISOMETRIC ACTIONS

- ACTIONS ON BANACH SPACES

((T) AND HAAGERUP CAN BE DEFINED USING ACTIONS ON HILBERT SPACES).

- ACTIONS ON SIMPLICIAL COMPLEXES

(BY SIMPLICIAL ISOMORPHISM)

→ THIS ALLOWS TO CONSTRUCTS NEW EXAMPLES: ∃ COND. ON THE LINKS OF SIMPLICIAL COMPL. ⇒ (T).

- LIMITS OF ACTIONS

② REPRES. BY ISOM. OF NON-POSITIVELY CURVED SPACES:

NPC: — THE ONE FROM RIEMANNIAN GEOM.
(SECTIONAL CURVATURE $K \leq 0$)

— METRIC GEN. OF THIS: CAT(0) SPACES.

NPC IS INTERTWINED WITH THE 2 PROP.:

- ALL KNOWN EX. COME WITH A NPC-STRUCTURE

- A CONCEPTUAL CONNECTION:

DELORME & GUICHARDET ; AKEMANN-WALTER:
FORMULATED EQ. DEF. OF (T) / a-(T)
USING ACTIONS ON SPACES NPC OF "MEDIAN TYPE"
(E.G. TREES, SIMPL. AND REAL, CAT(0) CUBE COMPLEXES)

## 3 ACTIONS BY LINEAR ISOM. ON BANACH SPACES

- IT IS THE INITIAL DEF. OF (T)
- THIS APROACH MORE APROPRIATE FOR:
  - CONSTRUCTING EXPANDERS
  - CONNECTING THE 2 PROP. TO $C^*$-ALGEBRAS, IDEMPOTENTS, B-C. CONJECTURE

# A GENERALIZED NOTION OF LIMIT

USE : – TO CONNECT APPROACHES 1 AND 2

– TO STUDY SPACES OF REPRES.

· WE NEED THE NOTION OF

NON-PRINCIPAL ULTRAFILTER.

· IT CAN BE SEEN AS FOLLOWS:

→ CONSIDER A SEQUENCE OF SEQUENCES: $A_n = \left(a_k^{(n)}\right)$ IN $[0,1]$. WE CAN:

→ TAKE A CONV. SUBSEQ OF $A_1$,

→ TAKE A SUBSEQ. OF PREVIOUS AND $A_2$, BOTH CONV.

⋮

USING DIAGONAL PROCEDURE WE FIND $(i_n) \subseteq \mathbb{N}$ s.t. EVERY $\left(a_{i_n}^{(n)}\right)$ CONV.

IF WE HAVE AN UNCOUNTABLE FAMILY OF SETS, WE NEED "THE" NON-PRINCIPAL ULTRAFILTER.

**DEF.** FIX $I$ AN INFINITE SET.

A FILTER $\mathcal{F}$ ON $I$ IS A FAMILY

$$\mathcal{F} \subseteq \mathcal{P}(I) \text{ S.T.}$$

(F1) $\emptyset \notin \mathcal{F}$

(F2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

(F3) $A \in \mathcal{F} \;\&\; A \subseteq A' \Rightarrow A' \in \mathcal{F}$.

Ex. ① THE SET OF NEIGHBOURHOODS OF A POINT $x$ IN A TOP. SPACE:

$A \subseteq X$ s.t. $x \in$ Open $\subseteq A$.

② $\mathcal{F} = \{ I \setminus F \,;\, F \text{ finite} \}$.

- THE CO-FINITE TOPOLOGY.
- ALSO CALLED THE FRECHET FILTER.

<u>DEF.</u> AN ULTRAFILTER $\mathcal{U}$ ON $I$ IS A FILTER MAXIMAL WRTO "$\subseteq$".

$\Updownarrow$

$\mathcal{U}$ ULTRAFILTER $\Leftrightarrow$ (F1) – (F3) AND

(F4) $\forall A \subseteq I$ $A \in \mathcal{U}$ OR $I \setminus A \in \mathcal{U}$.

<u>EXAMPLE</u> $\forall x \in I$

$$\mathcal{U} = \{ A \subseteq I ,\ x \in A \}$$

CALLED ATOMIC/PRINCIPAL ULTRAFILTER

<u>EXERCISE</u> PROVE $\exists$ AN ULTRAFILTER ON $\mathbb{Z}$ CONTAINING ALL NON-TRIVIAL SUBGROUPS.

(PROFINITE ULTRAFILTER).

WE ARE INTERESTED IN NON-PRINCIPAL $U$.

<u>LEMMA</u> AN ULTRAFILTER $\mathcal{U}$ IS NON-PRINCIPAL $\iff$ $\mathcal{U}$ CONTAINS THE FRECHET FILTER.

PROOF PRINCIPAL $\Leftrightarrow$ FRECHET $\not\subseteq \mathcal{U}$.

$\boxed{\Rightarrow}$ $\mathcal{U} = \{ A ; x \in A \}$

$\{x\} \in \mathcal{U} \Rightarrow I \setminus \{x\} \notin \mathcal{U}$

$\in$ Frechet $\mathcal{F}$

$\boxed{\Leftarrow}$ $\exists\, I \setminus F$, $F$ finite, $I \setminus F \notin \mathcal{U} \Rightarrow$

$\underset{(F4)}{\Rightarrow} F \in \mathcal{U}$.

TAKE $F \cap \bigcap_{A \in \mathcal{F}} A$.

IF EMPTY THEN $F = \{ x_1, .., x_n \}$

$\forall i, \exists A_i \not\ni x_i, \ A_i \in \mathcal{U}$

$F \cap A_1 \cap .. \cap A_n = \emptyset \notin \mathcal{U}$

CONTRADICT (F2)

THUS $\exists\, a \in F \cap \bigcap_{A \in \mathcal{U}} A \Rightarrow \mathcal{U}_a \subseteq \mathcal{U}$

$\Rightarrow \mathcal{U}_a = \mathcal{U}.$

$\boxed{\text{QED}}$

<u>REM.</u> THIS LEMMA ENSURES $\exists$ OF NON-PRINCIPAL U., PROVIDED WE ADMIT

ZORN AXIOM $\Leftrightarrow$ AX. OF CHOICE

## AN EQUIVALENT WAY OF DEF. ULTRAF.:

**DEF.** AN ULTRAFILTER ON $I$ IS A FINITELY ADDITIVE MEASURE

$$\omega : \mathcal{P}(I) \to \{0,1\}, \quad \omega(I) = 1.$$

**LEMMA** $\omega$ CAN BE WRITTEN AS A CHARACT. FUNCTION $\omega = \mathbb{1}_{\mathcal{U}}$, $\mathcal{U} \subseteq \mathcal{P}(I)$.

$\omega$ ULTRAFILTER $\Leftrightarrow$ $\mathcal{U}$ ULTRAFILTER.

<u>LEMMA</u> $\mathcal{U}$ NON-PRINCIPAL $\Leftrightarrow \omega(\text{Finite}) = 0$.

<u>REMARKS</u> ① $\omega(A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n) = 1$

$\Rightarrow \exists\, i_0 \in \{1, \ldots, n\}$ s.t. $\omega(A_{i_0}) = 1$

$\omega(A_i) = 0, \ \forall i \neq i_0$.

② $\omega(A) = 1$ AND $\omega(B) = 1 \Rightarrow \omega(A \cap B) = 1$.

TERMINOLOGY: IF $I_P = \{ i \in I ;$ A CERTAIN COND. P IS SATISFIED$\}$ IS S.T. $\omega(I_P) = 1$, WE SAY P OCCURS $\omega$-ALMOST SURELY.

<u>Ex.</u> TAKE $(x_n), (y_n) \subseteq \mathbb{R}$, $\omega$ ULTRAFILTER ON $\mathbb{N}$, $x_n = y_n$ $\omega$-a.s $\Leftrightarrow$

$$\Leftrightarrow \omega(\{ n ; x_n = y_n \}) = 1 .$$

WE NOTICED WE NEED AX. OF CHOICE FOR $\exists$ OF NON-PRINCIPAL ULTRAFILTERS.

- IN FACT WE USE A STRICTLY WEAKER ASSUMPTION: THE HAHN-BANACH EXTENSION THM.

WE SHALL DISPLAY AND USE A WEAK VERSION OF CONVERSE:

WE PROVE $\exists$ OF NON-P.U. $\Rightarrow$ HAHN-BANACH IN A PARTICULAR CASE:

IN THIS CASE: TAKE AS AMBIENT V.S.

$$V = \{ (x_n) \subseteq \mathbb{R} ; \text{ BOUNDED} \}.$$

$$W \leq V, \quad W = \{ (x_n) \text{ CONVERGENT} \}$$

THE FUNCTIONAL TO EXTEND

$$p : W \to \mathbb{R}, \quad p((x_n)) = \lim_{n \to \infty} (x_n).$$

GIVEN $\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ N.P.U. (NON-PRINCIPAL ULTRAFILTER), WE CAN DEFINE AN EXTENSION OF $p$ TO $V$.

DEF. GIVEN $(x_n) \in \mathbb{R}$ BOUNDED, WE SAY $a \in \mathbb{R}$ IS AN ω-LIMIT OF $(x_n)$ IF:

$$\forall U \subseteq \mathbb{R},\ a \in U \text{ open},$$

$$\omega\left(\left\{n \in \mathbb{N},\ x_n \in U\right\}\right) = 1.$$

<u>EX.</u> ① IF $x_n \longrightarrow x$, $x = \omega$-LIMIT OF $(x_n)$

② IF WE TOOK $\omega$ PRINCIPAL $\Leftarrow$

$\Leftarrow$ $\omega = \delta_k$ for some $k$, BY PREVIOUS DEF., $\omega$-limit $(x_n) = x_k$.