=1=

REMINDER:

$I \neq \phi$, I INFINITE

FILTER $\mathcal{F} \subseteq \mathcal{P}(I)$ S.T.

(F1) $\phi \notin \mathcal{F}$;

(F2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(F3) $A \in \mathcal{F}$, $A \subseteq B \Rightarrow B \in \mathcal{F}$.

ULTRAFILTER $\mathcal{U}$ = FILTER MAX. WRTO $\subseteq$.

EQUIVALENTLY, FILTER + (F4) $\forall A \subseteq I$, $A \in \mathcal{U}$ OR $I \setminus A \in \mathcal{U}$

=2=

FRÉCHET FILTER = $\Phi = \{ I \setminus F ; F \text{ FINITE} \}$.

$\mathcal{U}$ PRINCIPAL ULTRAFILTER = $\exists x \in I$:

$\mathcal{U} = \{ A \subseteq I , x \in A \}$.

$\mathcal{U}$ NON-PRINCIPAL $\iff \Phi \subseteq \mathcal{U}$.

SECOND DEF: A NON-PRINCIPAL ULTRAFILTER is $\omega : \mathcal{P}(I) \to \{0,1\}$ FIN. ADDITIVE MEAS.

CONN.: $\omega = \mathbb{1}_{\mathcal{U}}$.

= 3 =

DEF. GIVEN $f: I \to X$, X TOP. SP.
AN ω-LIMIT OF f IS $x \in X$ S.T.
$\forall\, U$ OPEN, $x \in U$,

$$\omega\big(\{ i \in I,\ f(i) \in U \}\big) = 1$$

=4=

LEMMA (1) IF X IS COMPACT THEN $\forall f : I \to X$ HAS AN ω-LIMIT. (FOR ANY ω N.P.U.)

(2) IF X IS HAUSDORFF THEN THE ω-LIMIT IS UNIQUE.

PROOF. (1) ARGUE BY CONTRAD., ASSUME f DOES NOT HAVE AN ω-LIMIT: $\forall x \in X$, $\exists U_x$ OPEN s.t. $\omega(f^{-1}(U_x)) = 0$.

= 5 =

$$X = \bigcup_{x \in X} U_x \ , \ X \text{ CPCT.} \Rightarrow$$

$$\Rightarrow X = \bigcup_{i=1}^{n} U_{x_i} \quad \text{FOR SOME } x_1, \dots, x_m \in X.$$

$$f^{-1}(X) = I = \bigcup_{i=1}^{n} f^{-1}(U_{x_i}).$$

APPLY ω: $1 = \omega(I) \leq \sum_{i=1}^{n} \omega\left(f^{-1}(U_{x_i})\right) = 0$ ※

② ASSUME $\exists\, x \neq y$ DISTINCT ω-LIM.

$\Rightarrow \exists\, U, V$ OPEN, $U \cap V = \phi$, $x \in U$, $y \in V$

x ω-LIM. $\Rightarrow \omega\left(f^{-1}(U)\right) = 1$

y ω-LIM. $\Rightarrow \omega\left(f^{-1}(V)\right) = 1$

$f^{-1}(U) \cap f^{-1}(V) = \phi$ ※

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=6=

REM. $x = \lim_\omega f \in \overline{f(I)}$.

$\forall\, U$ OPEN, $x \in U$, $\omega(f^{-1}(U)) = 1 \Rightarrow f^{-1}(U) \neq \emptyset$

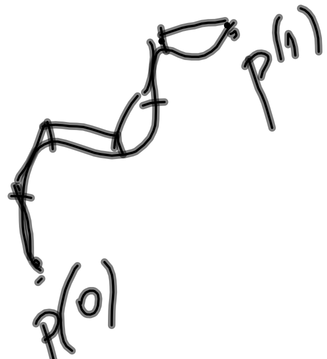
$\Rightarrow U \cap f(I) \neq \emptyset$.

LIMITS OF METRIC SPACES

PRELIMINARIES ON METRIC SPACES:

(X, d) METRIC SPACE

PATH $p : [0,1] \to X$



LENGTH $p =$

$$\mathrm{SUP}\left\{ \sum_{i=0}^{n-1} d(p(t_i), p(t_{i+1})) \;;\; \forall\, t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1 \right\}$$

=?=

GIVEN $x, y \in X$, A GEODESIC = A PATH

$p : [0,1] \to X$, $p(0) = x$, $p(1) = y$,

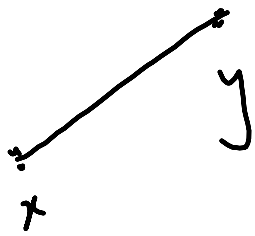
S.T. LENGTH $p = d(x,y)$.

X IS GEODESIC = $\forall\, x, y$, $\exists$ A GEODESIC

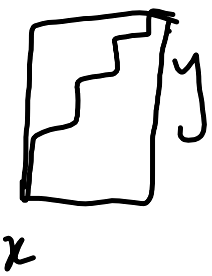
JOINING x, y

EX. ① $(\mathbb{R}^n, \|\cdot\|_e)$

$\|\cdot\|_2$



② $(\mathbb{R}^2, \|\cdot\|_1)$



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③ $(\mathbb{H}^2_{\mathbb{R}}, d)$

GIVEN G FINITELY GEN. GP., WE CAN ASSOCIATE A GEOD. METRIC SP. TO IT:

TAKE S FINITE SET GENERATING G.

$1 \notin S$, $S^{-1} = S$

WE CONSTRUCT THE CAYLEY GRAPH

CAY$(G;S)$: • VERTICES $V = G$

• EDGES $\{g, gs\}$, $g \in G$, $s \in S$.

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CAY(G, S) CONN. $\Leftrightarrow$ $\langle S \rangle = G$.

WE DEFINE d_S BY ASSUMING EDGES OF LENGTH 1

$\cdot\ d_S(x, y) =$ MIN. LENGTH OF A PATH JOINING THEM.

d_S IS CALLED WORD METRIC.

THM. OF HOPF-RINOW

A COMPLETE GEODESIC M.S. IS LOCALLY COMPACT IFF IT IS PROPER (I.E. ALL $\overline{B}(x, R)$ ARE COMPACT)

=10=

THM. OF MILNOR-SCHWARTZ:

IF A COUNTABLE GROUP G ACTS ON (X, d) COMPLETE, GEOD., LOC. COMPACT, BY ISOMETRIES, PROPERLY DISC. AND S.T. $G \backslash X$ COMPACT THEN:

1. G IS F.G.
2. $\forall S$ FINITE AS BEFORE, $\langle S \rangle = G$, (G, d_S) IS QUASI-ISOMETRIC TO (X, d).

= 11 =

DEF. A QUASI-ISOMETRY BETWEEN (X, d_X), (Y, d_Y) IS A MAP

$g: X \to Y$ S.T. $\exists\, L \geq 1,\ C \geq 0$:

- $\frac{1}{L} d(x_1, x_2) - C \leq d_Y(g(x_1), g(x_2)) \leq L\, d(x_1, x_2) + C$

- $Y \subseteq \mathcal{N}_C(g(X))$:

$\forall\, y \in X,\ \exists\, x \in X$ s.t. $d(y, g(x)) \leq C$.

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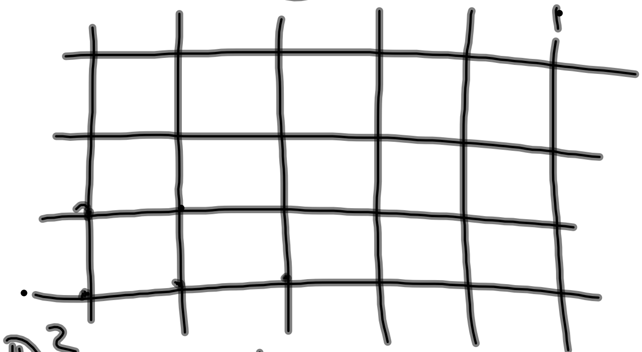
Ex. 1 $G = \mathbb{Z}^2 = \langle \underbrace{(\pm 1, 0), (0, \pm 1)}_{S} \rangle$

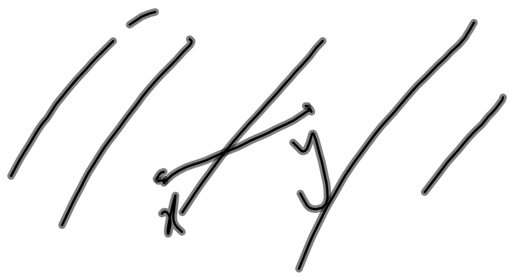
CAY(G, S):



$\mathbb{Z}^2$ ACTS ON $\mathbb{R}^2$ AS IN MILNOR-SCHWARTZ.

$\mathbb{R}^2, \|x-y\|_2$

$\mathbb{Z}^2$ Q.I. TO



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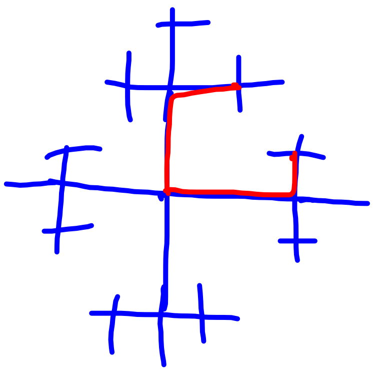
EX 2 $\mathbb{F}_2 = \langle a, b \rangle$

$\mathrm{CAY}(\mathbb{F}_2)$



ULTRALIMITS OF METRIC SPACES

CONSIDER A SEQ. (X_n, d_n).

CONSIDER ω N.P. ULTRAFILTER (N.P.U.)

$$\prod_{n \in \mathbb{N}} X_n = \{ (x_n) ; x_n \in X_n \}$$

$\forall\, x = (x_n),\ y = (y_n)$

TAKE $n \longmapsto d(x_n, y_n)$.

=14=

WE CAN DEFINE $\lim_\omega d_n(x_n, y_n) \in [0, +\infty]$.

$$\underbrace{\lim_\omega d_n(x_n,y_n)}_{d_\omega(x,y)}$$

TWO PROBLEMS:

① $d_\omega(x,y)$ MAY BE $+\infty$.

② $d_\omega(x,y) = 0 \not\Rightarrow (x_n) = (y_n)$.

① SOLVED BY CHOOSING BASEPOINTS $e_n \in X_n$.

WE RESTRICT TO

$$\prod_n^e X_n = \{ (x_n) ; \ (d(e_n, x_n)) \text{ IS BOUNDED} \}$$

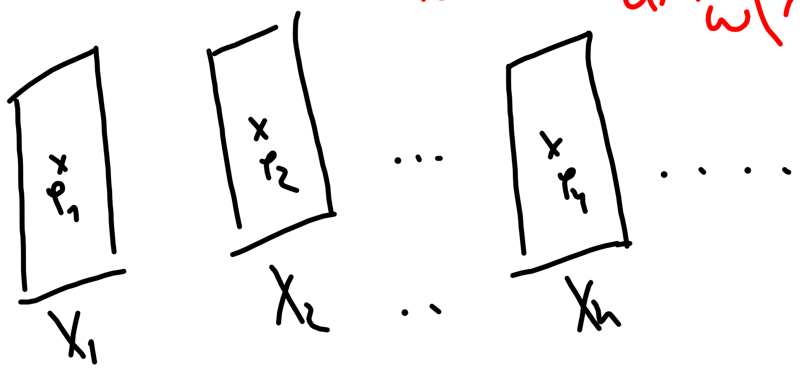
=14=

2 SOLVED BY CONSIDERING THE QUOTIENT:

$$\prod_n^{\ell} X_n \big/ \sim \;, \quad (x_n) \sim (y_n) \iff$$

$$\iff \lim_\omega d(x_n, y_n) = 0.$$

WE DENOTE THE ABOVE $\lim_\omega (X_n, d_n, e_n)$.



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A PARTICULAR CASE OF THIS LIMIT CONSTR.:

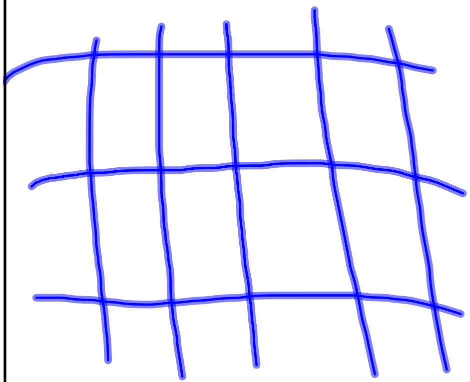
GIVEN ONE M.SP. (X, d)

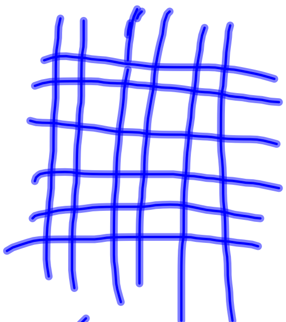
- WE CHOOSE $\lambda_n \in (0,+\infty)$, $\lambda_n \longrightarrow +\infty$
- WE CHOOSE $(x_n) \in X^{\mathbb{N}}$.

$\lim_\omega \left(X, \frac{1}{\lambda_n} d, x_n\right)$ IS CALLED

ASYMPTOTIC CONE OF X W.R.TO (λ_n) AND (x_n).

EXAMPLE (2) $G=\mathbb{Z}^2$, $S=\{(\pm 1, 0),(0,\pm 1)\}$,
$\lambda_n \longrightarrow +\infty$, $e_n=(0,0)$.



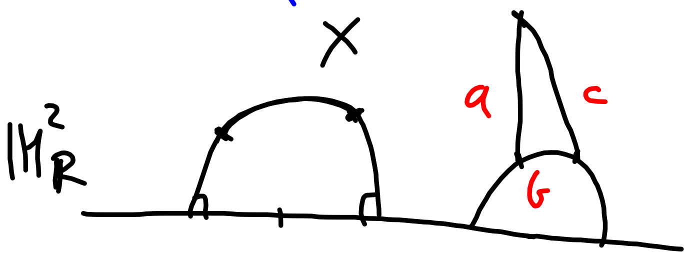


$\longrightarrow \left(\mathbb{R}^2, \| \ \|_1\right)$

STEP n

=17=

EX ② $(\mathbb{H}^2_{\mathbb{R}}, d)$. FOR EVERY GEOD. Δ



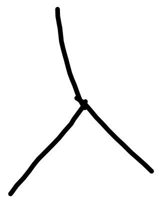
$a \in \mathcal{N}_\delta(b \cup c)$,

$\delta = \ln 2$

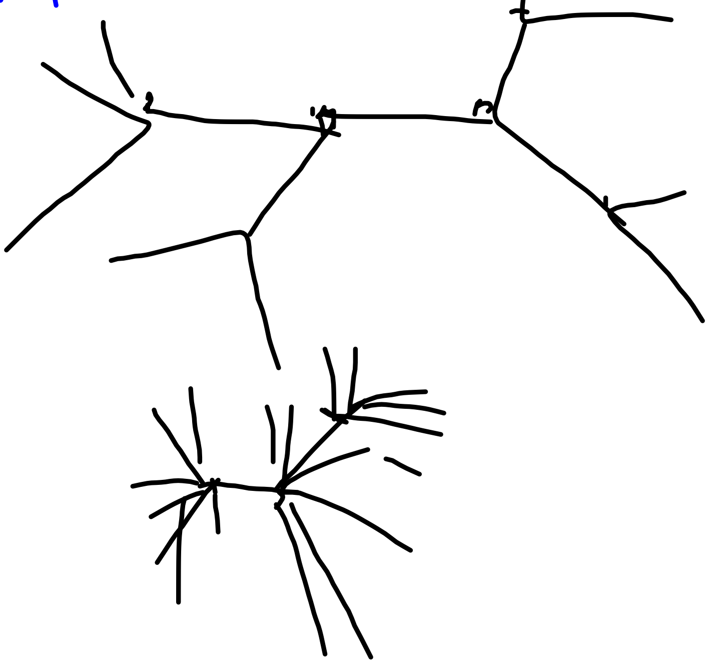
$\left(X, \frac{1}{\lambda_n} d\right)$, $a \in \mathcal{N}_{\frac{\delta}{\lambda_n}}(b \cup c)$

$\downarrow$

IN THE AS. CONE, $\forall$ GEODESIC Δ IS A TRIPOD.



I.E. THE AS. CONE IS A REAL TREE.



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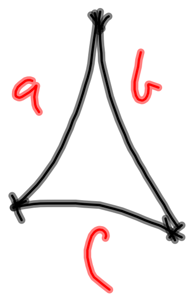
③ LET G A FIN. GEN. GROUP.

DEF. G IS GROMOV HYPERBOLIC IF FOR SOME (EQUIVALENTLY, FOR EVERY) CAYLEY GRAPH CAY(G, S), $\exists\, \delta > 0$ S.T. $\forall$ GEODESIC TRIANGLE IN IT:



$$a \subseteq N_\delta(b \cup c).$$

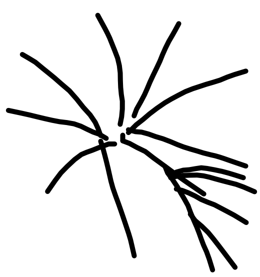
AS BEFORE, $\forall$ AS. CONE OF SUCH G IS A REAL TREE.

=19=

IN FACT, FOR $X = \mathbb{H}^n_{\mathbb{R}}$ OR $Cay(G,S)$, G GROMOV HYP. (NON-ELEMENTARY i.e. NO CYCLIC SUBGROUP OF FINITE INDEX): EVERY AS. CONE IS THE UNIVERSAL REAL TREE: EVERY REAL TREE (= GEODESIC M.S. WITH ALL Δ TRIPODS) WITH CARDINALITY $\leq 2^{\aleph_0}$ CAN BE ISOMETRICALLY EMBEDDED IN IT.

$\Updownarrow$

A COMPLETE REAL TREE S.T. EVERY POINT IS BRANCHING WITH $2^{\aleph_0}$ DIRECTIONS GOING OUT OF IT.



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BASIC PROP. OF ULTRALIMITS

① IF (X_n, d_n) ARE GEODESIC THEN EVERY $\lim_\omega(X_n, d_n, e_n)$ IS GEODESIC.

PROOF – EX.

② $\forall \omega$ N.P.U., $\forall (e_n)$, $\lim_\omega(X_n, d_n, e_n)$ IS COMPLETE.

PROOF STEP 1 $Y_n \subseteq X_n$ S.T. $\overline{Y_n} = X_n$

THEN $\lim_\omega(Y_n, d_n, e_n) = \lim_\omega(X_n, d_n, e_n)$, $e_n \in Y_n$.

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ALWAYS AN EMBEDDING

$$\lim_\omega (Y_n, \dots) \longrightarrow \lim (X_n, \dots)$$

$$\lim_\omega y_n \longmapsto \lim_\omega y_n$$

THE MAP IS ONTO BECAUSE $\forall \lim_\omega x_n$, $\exists\, y_n \in Y_n$ S.T. $d_n(x_n, y_n) < \frac{1}{n} \Rightarrow$

$\Rightarrow \lim_\omega x_n = \lim y_n$.

AS A CONSEQ., WE MAY REPLACE X_n BY THEIR METRIC COMPLETION.

THUS, WLOG X_n ARE COMPLETE

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STEP 2 CONSIDER $X_\omega = \lim_\omega (X_n, d_n, e_n)$.

TAKE $x^{(k)}$ CAUCHY IN X_ω.

BY EVENTUALLY TAKING A SUBSEQ.,

WE MAY ASSUME $\forall k \in \mathbb{N}$,

$$(*)\ d_\omega(x^{(k)}, x^{(k+1)}) < \frac{1}{2^k}.$$

EACH $x^{(k)} = \lim_\omega x_n^{(k)}$, $x_n^{(k)} \in X_n$.

$$(*)\ \lim_\omega d_n(x_n^{(k)}, x_n^{(k+1)}) < \frac{1}{2^k}$$

$\Rightarrow$ ω-ALMOST SURELY

$$d_n(x_n^{(k)}, x_n^{(k+1)}) < \frac{1}{2^k}.$$

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EQUIVALENTLY,

$$I_k = \left\{ n \in \mathbb{N};\ d_n\left(x_n^{(k)}, x_n^{(k+1)}\right) < \frac{1}{2^k} \right\}$$

$$\omega(I_k) = 1.$$

WLOG WE MAY ASSUME $I_{k+1} \subseteq I_k, \forall k,$
OTHERWISE REPLACE I_{k+1} BY $I_{k+1} \cap I_k$.

CASE 1 $\underbrace{\bigcap_{k \in \mathbb{N}} I_k}_{J}$ HAS ω-PROBA. 1.

$$\omega(J) = 1.$$

$$\forall n \in J,\quad d_n\left(x_n^{(k)}, x_n^{(k+1)}\right) < \frac{1}{2^k}.$$

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FIX $n \in \mathbb{J}$, LOOK AT SEQ. $\left(x_n^{(k)}\right)$.

$\forall\, m > k$, $d\left(x_n^{(k)}, x_n^{(m)}\right) \leq$

$\leq d\left(x_n^{(k)}, x_n^{(k+1)}\right) + \ldots + d\left(x_n^{(m-1)}, x_n^{(m)}\right) <$

$< \frac{1}{2^k} + \frac{1}{2^{k+1}} + \ldots + \frac{1}{2^{m-1}} < \frac{1}{2^{k-1}}$

$\Rightarrow$ THE SQ. $\left(x_n^{(k)}\right)$ IN X_n IS CAUCHY

X_n COMPLETE $\Rightarrow \left(x_n^{(k)}\right) \xrightarrow[k \to \infty]{} y_n$

WE HAD $\forall\, k < m$,

$$d_n\left(x_n^{(k)}, x_n^{(m)}\right) < \frac{1}{2^{k-1}}$$

FOR $m \to +\infty \Rightarrow$ $d_n\left(x_n^{(k)}, y_n\right) \leq \frac{1}{2^{k-1}}$.

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THUS WE OBTAINED $(y_n) \in \prod X_n$

S.T. $\forall n \in J$, $\forall k$, $d_n(x_n^{(k)}, y_n) \leq \frac{1}{2^{k-1}}$

$\Uparrow$

$d_n(x_n^{(k)}, y_n) \leq \frac{1}{2^{k-1}}$ IS TRUE ω-A.S.

$\Downarrow$

$d_\omega(x^{(k)}, y) = \lim_\omega d_n(x_n^{(k)}, y_n) \leq \frac{1}{2^{k-1}}$

$y = \lim_\omega (y_n)$

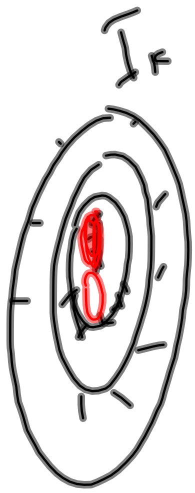
THUS, IN X_ω, FOR $(x^{(k)})$ CAUCHY, WE FOUND y S.T. $d_\omega(x^{(k)}, y) \le \frac{1}{2^{k-1}}$

$\Rightarrow x^{(k)} \longrightarrow y$.

CASE $\omega\Big(\underbrace{\bigcap_{k\in\mathbb{N}} I_k}_{y}\Big) = 0$.

$\omega(I_k) = 1, \ \forall k \in \mathbb{N}$.

$$I_k = y \cup \bigsqcup_{j\geq k} (I_j \setminus I_{j+1})$$



$$\Rightarrow \omega\Big(\bigsqcup_{j\geq k} (I_j \setminus I_{j+1})\Big) = 1.$$

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WE DEFINE $(y_n) \in \prod_{n \in I} X_n$ AS FOLLOWS:

$$y_n = e_n \quad \forall n \in (\mathbb{N} \setminus I_1) \sqcup J.$$

$$\forall n \in I_k \setminus I_{k+1}, \quad y_n = x_n^{(k)}.$$

WE COMPARE (y_n) TO $(x_n^{(k)})$.

$$\forall \underbrace{n \in I_k \setminus J}_{} = \bigsqcup_{j \geq k} (I_j \setminus I_{j+1})$$

$\Rightarrow \exists\ j \geq k$ s.t. $n \in I_j \setminus I_{j+1}$

BY DEF. OF (y_n), $y_n = x_n^{(j)}$.

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$n \in I_j \setminus I_{j+1} \subseteq I_{j-1} \subseteq \dots \subseteq I_k$

$\Rightarrow d(x_n^{(k)}, x_n^{(k+1)}) \leq \frac{1}{2^k}$

$d(x_n^{(k+1)}, x_n^{(k+1)}) \leq \frac{1}{2^{k+1}}$

$\vdots$

$d(x_n^{(j-1)}, x_n^{(j)}) \leq \frac{1}{2^{j-1}}$

$d(x_n^{(k)}, x_n^{(j)}) \leq \frac{1}{2^k} + \dots + \frac{1}{2^{j-1}} < \frac{1}{2^{k-1}}$

THUS $d(x_n^{(k)}, y_n) < \frac{1}{2^{k-1}}$.

BUT $\omega(I_k \setminus J) = 1$.

THUS WE OBTAINED THAT

$d_n(x_n^{(k)}, y_n) < \frac{1}{2^{k-1}}$ ω-A.S.

$\Downarrow$

$\lim_\omega d_n(x_n^{(k)}, y_n) =$

$= d_\omega(x^{(k)}, y) \leq \frac{1}{2^{k-1}}$.

-29-

THUS, FOR CHOSEN (y_n), $y = \lim_\omega (y_n)$

SATISFIES $d_\omega(y, x^{(k)}) \leq \frac{1}{2^{k-1}} \Rightarrow$

$\Rightarrow x^{(k)} \longrightarrow y$. $\boxed{QED}$

PROPERTIES OF ASYMPTOTIC CONES

GIVEN (X, d)

FOR A CHOICE OF $(\lambda_n) \in (0, +\infty)$, $\lambda_n \to \infty$,

$(p_n) \in X^{\mathbb{N}} \longrightarrow$ AN AS. CONE

$$X_\omega((\lambda_n), (p_n))$$

(C1) EVERY ULTRALIMIT OF AS. CONES OF X IS AN AS. CONE OF X.

$$C_m = X_{\omega_m}\left(\left(\lambda_k^{(m)}\right), \left(e_k^{(m)}\right)\right)$$

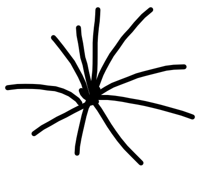
$\lim_\mu (C_m, d, x_m)$ IS A CONE FOR A GOOD CHOICE OF ULTRAFILTER, SCALARS, BASEPOINTS.

(C2) MOST ASYMPTOTIC CONES ARE NOT LOCALLY COMPACT.

IF G A F.G. GROUP WITH A WORD METRIC (G, d_S) HAS ALL AS. CONES LOCALLY COMPACT THEN G CONTAINS A FINITE INDEX SUBGROUP G_1, THAT IS NILPOTENT

31.

MOREOVER, ALL AS. CONES WILL BE ISOMETRIC.



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