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REMINDER & MORE PROOFS

CONDITIONALLY NEGATIVE DEFINITE KERNEL:

KERNEL: $\psi : X \times X \to [0, +\infty)$, $\psi(x,y) = \psi(x,y)$

$\psi(x,x) = 0$;

CND: $\forall n \in \mathbb{N}$, $\forall x_1, \dots, x_n \in X$,

$\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$, $\sum \lambda_i = 0$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi(x_i, x_j) \leq 0.$$

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<u>Ex. (MAIN):</u> $X = H$ HILBERT, $\psi(x,y) = \|x-y\|^2$.

$$\sum_{i,j} \lambda_i \lambda_j \|x_i - x_j\|^2 = \sum_{i,j} \lambda_i \lambda_j \big(\|x_i\|^2 + \|x_j\|^2 -$$

$$-2 \langle x_i, x_j \rangle \big) = \sum_i \lambda_i \|x_i\| \cdot \underbrace{\sum_{j=1}^{n} \lambda_j}_{} + \dots$$

$$\dots -2 \sum_{i,j} \lambda_i \lambda_j \langle x_i, x_j \rangle = -2 \Big\| \sum_{i=1}^{n} \lambda_i x_i \Big\|^2 \leq 0.$$

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(2) WELLS & WILLIAMS – "EMBEDDINGS & EXTENSIONS IN ANALYSIS", THM. 5.11:

If $1 \leq p \leq q \leq 2$ THEN ANY $L^p(\Omega, \mu)$ CAN BE EMBEDDED isometrically, WHEN ENDOWED WITH $\|x-y\|_p^{p/q}$, INTO SOME L^q-SPACE.

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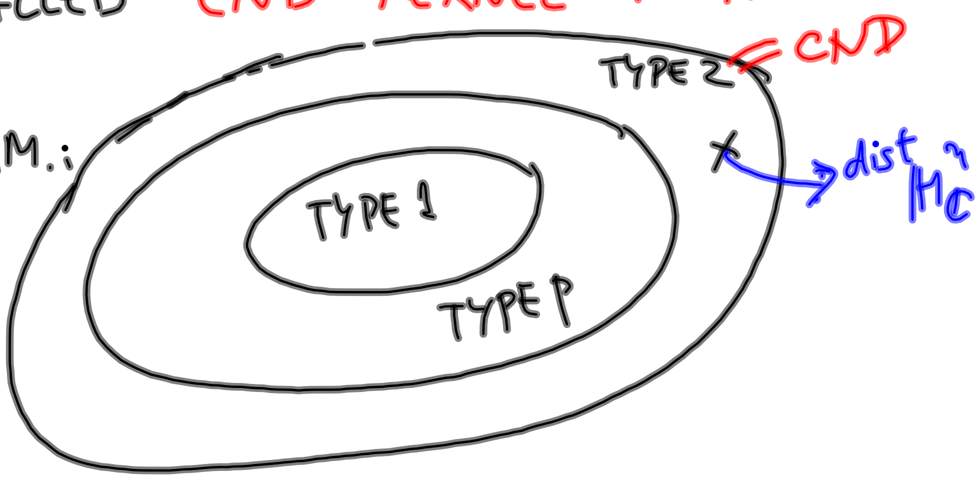
THUS $\|x-y\|_p^p$ IS CND.

IN GEN., GIVEN $f: X \to L^p$, $p \in [1,2]$,

$$\psi(x,x') = \|f(x)-f(x')\|_p^p$$

IS CALLED CND KERNEL OF TYPE P.

EMBEDDING THM.:



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THM. (SCHOENBERG, '53)

ψ CND KERNEL $\Leftrightarrow$ TYPE 2.

THM. (1) G HAS (T) $\Leftrightarrow$ $\forall\, \psi : G \times G \to [0, +\infty)$ CND KERNEL, $\psi(gx, gy) = \psi(x, y)$, ψ CONT. MUST BE BOUNDED.

(2) G HAS HAAGERUP $\Leftrightarrow$ $\exists\, \psi$ AS ABOVE, ψ PROPER.

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COR. (1) IF G HAS (T) THEN, $\forall\, p \in [1,2]$, G HAS FL^p: EVERY ACTION BY ISOM. ON AN L^p HAS A GLOBAL FIXED POINT.

(2) IF G IS A-FL^p-MENABLE: $\exists$ A PROPER ACTION ON AN L^p-SPACE THEN G IS A-T-MENABLE.

REM. THE CONVERSE ALSO TRUE: IF, FOR $p \in [1,2]$, G HAS FL^p THEN G HAS (T).

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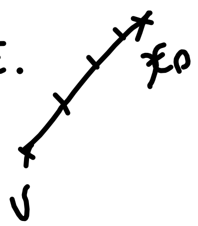
Ex.1 IF T IS A SIMPLICIAL TREE THEN d_T IS OF TYPE 1:

. FIX A VERTEX $x_0 \in V$

. DEFINE, GIVEN $\mathcal{E}$ SET OF EDGES:

$$f : V \longrightarrow \ell^1(\mathcal{E})$$
$$v \longmapsto \chi_{[x_0, v]}(e) = \begin{cases} 1, & e \leq [x_0, v], \\ 0 & \text{OTHERWISE.} \end{cases}$$



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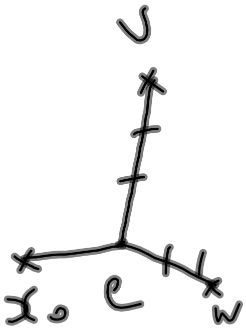
$$\|\chi_{[x_0,v]} - \chi_{[x_0,w]}\| = \|\chi_{[e,v]} - \chi_{[e,w]}\| = d_T(v,w).$$



EX. 2 X CAT(0) CUBE COMPLEX:

A COMPLEX · OBTAINED BY GLUEING $[0,1]^n$ ALONG FACES S.T.

· THE SHORTEST PATH METRIC ≡ EUCLIDEAN M. ON CUBE IS CAT(0)

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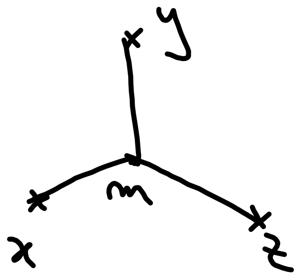
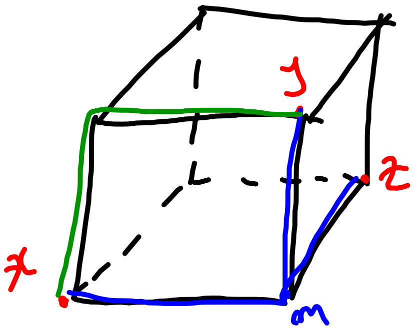
A FEW GEOMETRIC FEATURES OF X:

· TAKE $X^{(1)}$ 1-SKELETON + SHORTEST PATH METRIC.

$\forall x, y, z \in X^{(0)}$, $\exists$ A GEODESIC TRIANGLE:



NB1 UNLIKE FOR TREES,
$\exists$ SEVERAL GEOD. BETW. 2 POINTS



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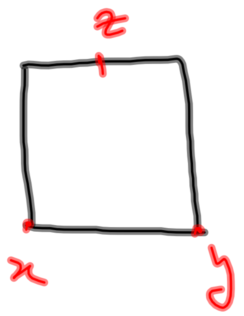
NB2 THE PROPERTY NO LONGER TRUE

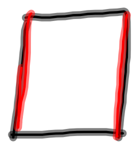
$\forall\, x, y, z \in X^{(1)}$.



THE CAT(0)-C.C. STR. ALLOWS TO CONSTRUCT CONVEX WALLS:

- DEFINE A REL. BETWEEN EDGES:

$e R e' \iff e, e'$ IN SAME CUBE, $e \parallel e'$



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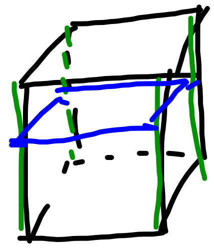
LET $\overline{R}$ ITS TRANSITIVE CLOSURE.

GIVEN $[e]$ EQUIV. CLASS FOR $\overline{R}$, CONSTRUCT A CORRESPONDING HYPERPLANE $= \bigcup$ ALL CUBES WITH VERTICES MIDPOINTS OF EDGES IN $[e]$:



CONSIDER $\mathcal{H}$ = ALL HYP., FOR ALL $[e]$.

- EVERY HYP. IS CONVEX, CAT(0).
- $\forall A, B$ DISJOINT, CONVEX SUBCOMPLEXES, $\exists$ HYP. SEPARATING THEM.

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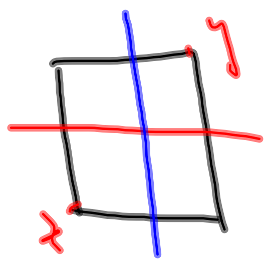
$\forall x, y \in V$, $d_{X^{(1)}}(x,y) = \#\{$HYPERPLANES SEPARATING $x, y\}$.

- $\forall$ GEODESIC IN $X^{(1)}$ JOINING x, y MUST CROSS ALL THESE HYP., NO OTHER. ORDER MAY DIFFER



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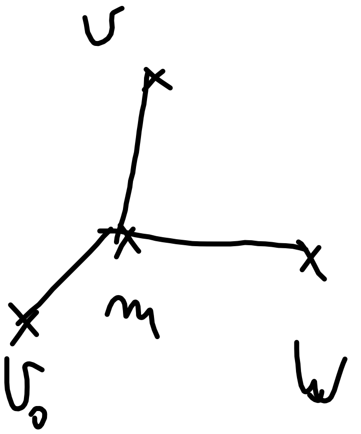
WE DEFINE $\overset{-B-}{V} \longrightarrow \ell^1(\mathcal{H})$

$$v \longmapsto \chi_{\{\text{hyperplanes separating } v \text{ from } x_0\}}$$

WHERE x_0 IS A FIXED BASEPOINT.



$$\left\| \chi_{\mathcal{H}(x_0,v)} - \chi_{\mathcal{H}(x_0,w)} \right\|$$

$$= \left\| \chi_{\mathcal{H}(m,v)} - \chi_{\mathcal{H}(m,w)} \right\| =$$

$$= d_{X^{(1)}}(v,w).$$

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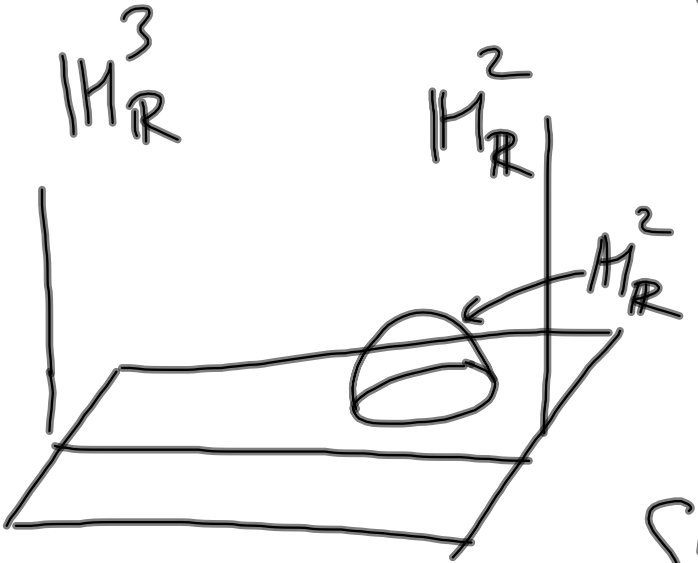
EX. 3 TAKE $X = \mathbb{H}^n_{\mathbb{R}} = SO(n,1)/SO(n)$



X CONTAINS $\mathbb{H}^{n-1}_{\mathbb{R}}$

$SO(n,1) = \mathrm{isom}(\mathbb{H}^n_{\mathbb{R}})$

ACTS TRANSITIVELY, STAB. $\mathrm{isom}(\mathbb{H}^{n-1}_{\mathbb{R}})$

THUS, THE SET OF HYP. HYPERPLANES IN $\mathbb{H}^n_{\mathbb{R}}$ CAN BE IDENTIF. WITH

$$SO_{Id}(n,1)/SO_{Id}(n-1,1)$$

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BECAUSE $SO_{Id}(n-1,1)$ IS UNIMODULAR

$\Rightarrow \exists$ AN $SO_{Id}(n,1)$-INVARIANT BORELIAN MEASURE μ ON $SO_{Id}(n,1)/SO_{Id}(n-1,1)$.

CROFTON FORMULA: $\exists \lambda \in (0,+\infty)$ S.T. $\forall x, y \in \mathbb{H}^n_{\mathbb{R}}$,

$$d_{\mathbb{H}^n_{\mathbb{R}}}(x,y) = \lambda \mu(\{\text{hyperplanes separating } x \text{ and } y\})$$

THUS $f: \mathbb{H}^n_{\mathbb{R}} \to L^1(SO(n,1)/SO(n-1,1), \mu)$

$$f(x) = \chi_{\{\text{hyp. separating } x \text{ and } x_0\}}.$$

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f isom. embedding $\Rightarrow d_{M_{\mathbb{R}}^{\gamma}}$ type 1.

Ex. 4 if X a real tree (geodesic, 0-hyp., ie all geod. Δ are tripods) then d_X of type 1.

$\Downarrow$

G has (T) $\Rightarrow$ G has FR:

$\forall$ action on a complete real tree has a fixed point.

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THE CONVERSE NOT TRUE. MUST TAKE A LARGER CLASS THAN REAL TREES.

COUNTER-EX. TO CONVERSE:

COXETER GROUPS:

$$G = \langle x_1, \dots, x_m \mid x_i^2 = 1,\ (x_i x_j)^{m_{ij}} = 1 \rangle.$$

$m_{ij} = \infty \iff$ NO RELATION.

HERE ASSUME ALL $m_{ij} < +\infty$.

IF $G \curvearrowright X$ REAL TREE:

- $\forall g$ OF FINITE ORDER HAS $\mathrm{Fix}(g) = \{x \,;\, gx = x\} \neq \emptyset$.

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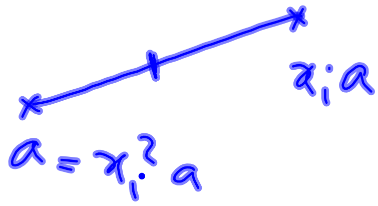
- $x_i^2 = id \Rightarrow \forall a \in T$, THE MIDPOINT OF $[a, x_i a] \in Fix(x_i)$



- $\forall\, m_{ij} < +\infty$

$\Rightarrow \exists\, a \in Fix(x_i x_j) \Rightarrow x_i x_j a = a$

$\Rightarrow x_j a = x_i a \Rightarrow$ MIDPOINT $[a, x_i a] \in Fix(x_i) \cap Fix(x_j)$.

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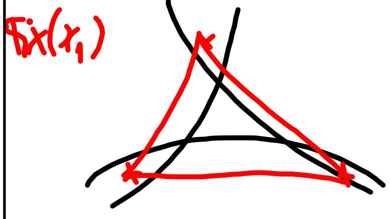
THUS $\{\mathrm{Fix}(\gamma_i),\ i \in \{1,\dots,n\}\}$ IS A COLL.

OF CONVEX SUBSET THAT $\cap$ PAIRWISE

$\Longrightarrow$ HELLY THM. $\bigcap_{i=1}^{n} \mathrm{Fix}(\gamma_i) \neq \emptyset$

CLEAR IF $n=3$:



→ REST BY IND. ON n.

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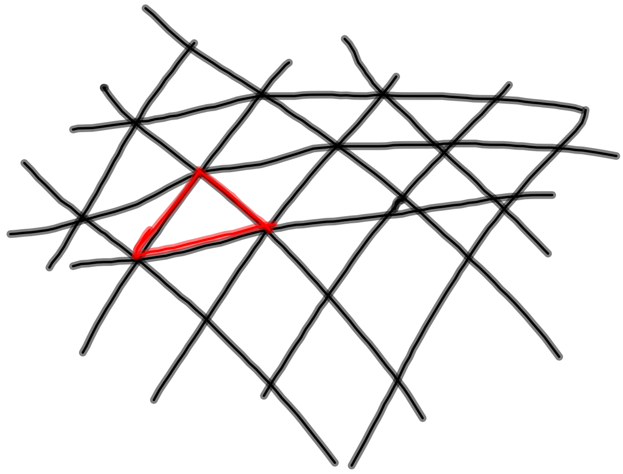
- ON THE OTHER HAND, ALL COXETER GROUPS ARE A-T-MENABLE.
- OBVIOUS FOR REFLEXION COXETER GPS.



$\mathbb{R}^2$

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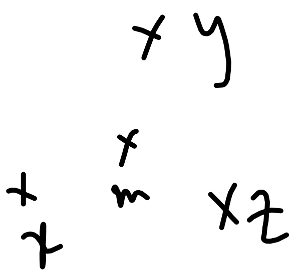
TO HAVE EQUIV.:

(X, d) MEDIAN IF $\forall x, y, z \in X$

$\exists\, m \in X$ S.T. m IS INBETWEEN x, y:

$$d(x,m) + d(m,y) = d(x,y);$$

· m INBETWEEN y, z; x, y



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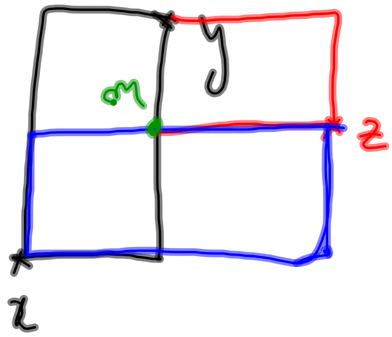
Ex. ① X REAL TREE;

② X CAT(0) C.C., $X^{(0)}$ with $d_{X^{(1)}}$.

③ $(\mathbb{R}^2, \| \ \|_1)$



④ $L^1(X, \mu)$ IS MEDIAN.

⑤ GIVEN X SIMPLICIAL GRAPHS

$X^{(0)}$ IS MEDIAN $\iff$ X IS THE 1-SKELETON OF A CAT(0) C.C.

(V. CHEPOI)

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IN COMPUTER SCIENCE, COMBINATORIAL OPTIMIZATION,

MEDIAN GRAPHS = GRAPHS REPRESENTING THE MOST ECONOMIC WAY OF SERVICING A GIVEN NUMBER OF REQUIREMENTS.

ANALOGIES TO HAVE IN MIND

SIMPLICIAL TREES $\longrightarrow$ REAL TREES

VERTICES OF CAT(0)-CC. $\longrightarrow$ NON-DISCRETE VERSION OF THESE: MEDIAN M.S.

WE DEFINED A "UNIVERSAL REAL TREE" = CONTAINING $\forall$ TREES OF $\leq 2^{\aleph_0}$.

A "UNIVERSAL MEDIAN SPACE":

$L^1[0,1]$, $L^1(X,\mu)$.

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THM- (1) G HAS (T) $\Leftrightarrow$ EVERY ACTION ON A MEDIAN SPACE HAS BOUNDED ORBITS.

(2) G IS A-T-MENABLE $\Leftrightarrow$ $\exists$ A PROPER ACTION ON A MEDIAN SPACE.

Ex. $\mathbb{H}^n_{\mathbb{C}} = SU(n,1)/SU(n)$.

- $d_{\mathbb{H}^n_{\mathbb{C}}}$ IS OF TYPE 2 (CND).

FARRAUT-HERZALLAH $\rightarrow$ CALCULATION

- $d_{\mathbb{H}^n_{\mathbb{C}}}$ NOT OF TYPE 1.

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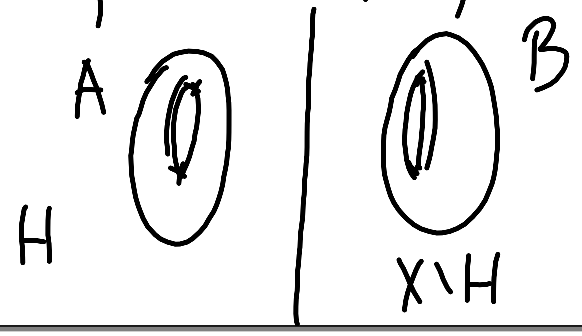
OTHERWISE, THERE $\exists\, f : \mathbb{H}^n_{\mathbb{C}} \to L^1(X,\mu)$, f ISOMETRIC EMBEDDING.

WELLS-WILLIAMS: IN $X = L^1(X,\mu)$, GIVEN A, B, $A \cap B = \emptyset$, A, B CONVEX $\exists\, H$ S.T. $A \subseteq H$, $B \subseteq X \setminus H$, $H, X \setminus H$ CONVEX



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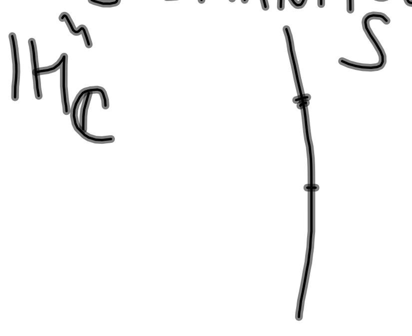
THEN, SAME MUST BE TRUE $\forall\, Y$ METRIC SP. ISOMETRICALLY EMBEDDABLE IN AN L^1.

$\Rightarrow$ SHOULD BE TRUE FOR $\mathbb{H}^n_{\mathbb{C}}$ $\Rightarrow$

$\Rightarrow$ $\exists\, S \subseteq \mathbb{H}^n_{\mathbb{C}}$, S CODIMENSION 1 SUBMANIFOLD THAT IS TOTALLY GEOD.



IMPOSSIBLE IN $\mathbb{H}^n_{\mathbb{C}}$.

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LIMIT ACTIONS AGAIN

CONSIDER 2 FINITELY GEN. GPS.

H, G, $H = \langle S \rangle$, S, X FINITE
$G = \langle X \rangle$, $S^{-1} = S$, $1 \notin S$ + X.

ASSUME WE HAVE A SEQUENCE OF GP. HOM. PAIRWISE NON-CONJ. IN G:

- $\varphi_n : H \to G$, $n \in \mathbb{N}$,
- $n \neq m$, $\nexists\ c_g : G \to G$, $x \mapsto g x g^{-1}$ s.t.

$$\varphi_n = c_g \circ \varphi_m .$$

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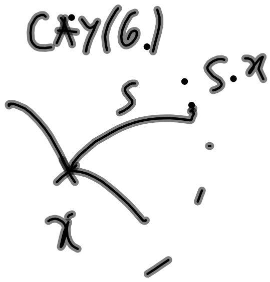
EACH φ_n DEFINES AN ACTION BY ISOM. OF H ON G:

$$h \cdot g = \varphi_n(h) \cdot g.$$

+ WORD METRIC

A RELEVANT PARAMETER = THE MINIMAL DISPLACEMENT:

$$\lambda_n = \min_{x \in G} \max_{s \in S} d(x, \varphi_n(s)x)$$



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(φ_n) PAIRWISE NON-CONJ. $\Rightarrow \lim_{n\to\infty} \lambda_n = +\infty$

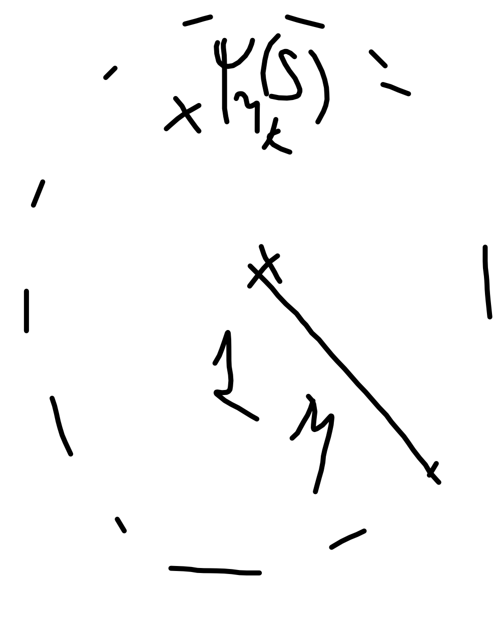
INDEED, IF $\exists\, (\lambda_{n_k})$ BOUNDED BY $M>0$

$\Rightarrow \forall k,\ \lambda_{n_k} \leq M \iff$

$\iff \exists\, x_{n_k}$ s.t. $d(x_{n_k}, \varphi_n(S)\, x_{n_k}) \leq M$

$\Updownarrow$

G $\quad d\big(c_{x_{n_k}} \circ \varphi_{n_k}(S),\ 1\big) \leq M$



TAKE

$\psi_{n_k} = c_{n_k} \circ \varphi_{n_k}$

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$B(1, M)$ IN G HAS FINITELY MANY ELEMENTS.

EVENTUALLY FOR A SUBSEQ. OF (φ_{n_k}), THE IMAGE OF EACH $s \in S \subseteq H$ IS THE SAME $\Rightarrow$

$\Rightarrow$ ALL ψ_{n_k} ARE IDENTICAL $\Rightarrow$

$\Rightarrow$ ALL φ_{n_k} ARE CONJUGATE.

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THUS (φ_n) NON-CONV. $\Rightarrow \lim \lambda_n = +\infty$.

$H \underset{\varphi_n}{\curvearrowright} G$ WITH MIN. DISPL. λ_n

TAKE $\mathrm{CONE}_\omega(G\,;\,1,(\lambda_n)) = C_\omega$

WILL HAVE AN ACTION BY ISOM. OF H,
WITH MIN. DISPLACEMENT 1.

(BESTVINA-PAULIN)

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THM. (PAULIN) IF H HAS (T) & G IS GROMOV HYP. THEN

$$\mathrm{HOM}(H,G)/\mathrm{CONJ}.(G) \text{ IS FINITE.}$$

IN PART., IF H HYP. + (T),

$$\mathrm{OUT}(H)=\mathrm{AUT}(H)/\mathrm{CONJ}(H) \text{ IS FINITE.}$$

PROOF. $\exists\ \varphi_n : H \to G$ NON-CONJ.

$\Rightarrow H \curvearrowright \mathrm{Cone}(G)$ = REAL TREE ~~WITH~~ MIN. DISPL. 1. #