-1-

LIMIT ACTIONS

$H = \langle S \rangle$, $G = \langle X \rangle$, S, X finite

- $\forall$ homom. $\varphi : H \to G$ defines an action by isom. of H on (G, d_X):

$$h \cdot g = \varphi(h) \cdot g$$

- MINIMAL DISPLACEMENT:

$$\lambda = \min_{g \in G} \max_{s \in S} d_X(g, \varphi(s)g)$$

-2-

. TAKE $\varphi_i : H \to G$ HOMOM. THAT ARE CONJUGATE (IN G): $\exists\, g \in G$ s.t.

$$\varphi_1 = C_g \circ \varphi_2 \,, \quad C_g(x) = gxg^{-1}.$$

METRICALLY, THE ACTIONS $H \curvearrowright G$ AND $H \curvearrowright_{\varphi_2} G$ ARE IDENTICAL: φ_1

THE ACTION OF $H \curvearrowright_{\varphi_2} G$ IN $\gamma \in G$ IS THE SAME METRICALLY AS

$$H \curvearrowright_{\varphi_1} G \text{ IN } g\gamma.$$

IN PART. SAME MIN. DISPLACEMENT.

-3-

· TAKE $\varphi_n : H \to G$, $n \in \mathbb{N}$, PAIRWISE NON-CONJ. (IF ∃).

WE PROVED THAT IF λ_n MIN. DISPL. OF $H \curvearrowright_{\varphi_n} G$ THEN $\lambda_n \to +\infty$.

· TAKE $x_n \in G$ WHERE λ_n IS REALIZED.

$H \curvearrowright_{\varphi_n} \left(G, \frac{1}{\lambda_n} d_X, x_n\right)$ ACTION WITH MIN. DISPL. 1.

-7-

LEMMA THE ACT. OF H ON $\mathrm{Cone}_\omega(G;(\lambda_n),(x_n))$ (H ω NPU) HAS MIN. DISPLACEMENT 1.

PROOF - EX.

THEOREM (PAULIN)

IF H HAS (T) AND G IS HYP. THEN

$\mathrm{Hom}(H,G)/\mathrm{Conj.}(G)$ IS FINITE.

IN PART. IF H HAS (T) & HYP. THEN $\mathrm{Out}(H)=\mathrm{Aut}(H)/\mathrm{Conj.}$ IS FINITE

-5-

PROOF if $\exists \varphi_n : H \to G$ NON-CONJ. $\Rightarrow$

(LEMMA) H ACTS ON SOME $CONE_\omega(G)$

WITH MIN. DISPL. 1.

COMPLETE REAL TREE

REMARK THIS IS, IN A VAGUE SENSE, AN "INCOMPATIBILITY RESULT": A GP. WITH (T') CANNOT HAVE ∞ MANY METRICALLY DISTINCT ACT. ON A GROMOV HYP. GP. G.

-6-

ANOTHER APPLICATION: TAKE G = A MAPPING CLASS GROUP OF A SURFACE.

TAKE S A CLOSED ORIENTED SURFACE, GENUS ≥ 2.

RMK WORKS ALSO WHEN $\partial S \neq \phi$.

THE MAPPING CLASS GP. OF S :

$$MCG(S) = HOMEO(S) / HOMEO_{id}(S)$$

HOMEOS ISOT. TO ID.

- FIN. GEN.
  (DEHN - LICHNEROWICZ).

-7-

THEOREM (BEHRSTOCK-D-SAPIR)

FOR EVERY AS. CONE OF $MCG(S)$,

$$X_\omega = \mathrm{CONE}_\omega\left(MCG(S), id, \lambda_n\right) + d_\omega$$

$\exists$ D_ω ANOTHER DISTANCE S.T.

- D_ω BI-LIPSCHITZ TO d_ω
- $\mathrm{ISOM}(X_\omega, d_\omega) \subseteq \mathrm{ISOM}(X_\omega, D_\omega)$.
- (X_ω, D_ω) IS MEDIAN SPACE, MOREOVER ITS GEODESICS ARE LIMITS OF HIERARCHY PATHS.

-8-

COR. IF H HAS (T) AND $G = MCG(S)$ THEN $\#Hom(H, G)/conj$ FINITE.

REM. $Out(MCG(S))$ IS FINITE, OTHERWISE A PROOF THAT G DOES NOT HAVE (T).

FOR THE MOMENT: ∃ A PROOF THAT $MCG(S)$ DOES NOT HAVE (T), DUE TO). ANDERSON, STILL BEING CHECKED.

-9-

* GENERAL CONSTR. OF LIMIT ACTIONS

- DUE TO GROMOV ("RANDOM WALK THROUGH RANDOM GROUP")
- IMPLIES 2 RESULTS:
- SHALOM'S THEOREM: $\forall$ G F.G. WITH (T) IS THE QUOTIENT OF A F.P. GROUP WITH (T).
- THE FIXED POINT PROPERTY IS OPEN, IN A TOPOLOGICAL SENSE.

– 10 –

TAKE G F.G., $G = \langle S \rangle$, S FINITE

TAKE INCREASING SEQ. OF $N_n \triangleleft G$, $N_n \subseteq N_{n+1}$,

- $G_n = G/N$
- THE DIRECT LIMIT $G_\infty = \varinjlim G_n$.

EX. G_∞ = A F.G. GP., NOT F.P.

$G_\infty = \langle S \mid R_1, \dots, R_m, \dots \rangle$

$G = F_S$, N_n = NORMAL CLOSURE OF $\{R_1, \dots, R_n\}$,

$G_n = \langle S \mid R_1, \dots R_n \rangle$.

-11-

CONVENTION: IMAGE OF S IN G_n DENOTED S.

THEOREM. ASSUME G_n HAS AN ACTION ON A COMPLETE M.S. WITH BASEPOINT

$$(X_n, d_n, x_n),$$

WITHOUT GLOBAL FIXED POINT THEN

$\forall\ \delta_n > 0$, $\delta_n \to \infty$, $\exists\ y_n \in X_n$ AND

$$\lambda_n \geq \frac{2}{(1+4\delta_n)\cdot \mathrm{DIAM}(S_{x_n})}$$ S.T.

$$G_\infty \curvearrowright \lim_\omega (X_n, \lambda_n d_n, y_n)$$

WITHOUT FIXED POINT.

PROOF STEP 1 Fix ω NPU.

WE PROVE THAT ω-A.S. $\exists\, y_n \in B(x_n, 2\delta_n \mathrm{DIAM}(S_{x_n}))$

S.T. $\forall\, z_n \in B\left(y_n, \dfrac{\delta_n \mathrm{DIAM}(S_{y_n})}{2}\right)$,

$$\mathrm{DIAM}(S_{z_n}) \geq \frac{\mathrm{DIAM}(S_{y_n})}{2}$$

-13-

ASSUME NOT TRUE : ω-A.S.

$$\forall y \in B(x_n, 2\delta_n \operatorname{diam}(Sx_n))$$

$$\exists\, z \in B\left(y, \frac{\delta_n \operatorname{diam}(Sy)}{2}\right) \text{ S.T.}$$

$$\operatorname{diam}(Sz) < \frac{\operatorname{diam}(Sy)}{2}$$

HENCE $\{n \in \mathbb{N};$ THE ABOVE TRUE$\}$ IS

$\omega(\{\dots\}) = 1 \Rightarrow \{\dots\}$ INFINITE.

-14-

FIX n IN THE SET, IE. WE FOCUS ON $G_n \curvearrowright X_n$, WE ARE GOING TO CONTRADICT "NO FIXED POINTS" (X_n COMPLETE).

WE DROP INDEX n.

WE TAKE $x_0 = x = x_n$.

WE APPLY THE PROPERTY TO $y = x$.

NOTATION: $D = 2\delta_n \operatorname{DIAM}(S_x)$

$R = \frac{D}{2}$.

$\exists\ z_1 \in B\left(x_0, \frac{R}{2}\right)$ S.T.

$$\operatorname{DIAM}(Sz_1) < \frac{\operatorname{DIAM}(Sx_0)}{2}$$

TAKE $x_1 = z_1$.

-15-

APPLY PROP. TO u_1 :

$$\exists z_2 \in B\left(u_1, \frac{\delta \operatorname{DIAM}(S_{u_1})}{2}\right) \subseteq B\left(u_0, R/2 + R/4\right)$$

$$\operatorname{DIAM}(S_{z_2}) < \frac{\operatorname{DIAM}(S_{u_1})}{2}$$

TAKE $u_2 = z_2$.

BY IND. WE CONSTR. $u_0, u_1, \ldots, u_k$ S.T.

$$u_i \in B\left(u_0, \frac{R}{2} + \ldots + \frac{R}{2^i}\right)$$

$$\operatorname{DIAM}(S_{u_i}) < \frac{\operatorname{DIAM}(S_{u_{i-1}})}{2} <$$

$$< \frac{\operatorname{DIAM}(S_{u_0})}{2^i}$$

$$u_i \in B\left(u_{i-1}, \frac{\delta \operatorname{DIAM}(S_{u_{i-1}})}{2}\right)$$

$$\Rightarrow d(u_i, u_{i-1}) < \frac{\operatorname{DIAM}(S_{u_0})}{2^i}$$

-16-

WE APPLY PROP. TO U_k, FIND

$$U_{k+1} \subset B\left(U_k, \frac{\delta \operatorname{DIAM}(S_{U_k})}{2}\right) \subseteq B(U_0, \dots)$$

$$\operatorname{DIAM}(U_{k+1}) < \frac{\operatorname{DIAM}(U_k)}{2}.$$

WE CONCLUDE THAT $\exists (U_k)$ S.T.

- $d(U_k, U_{k+1}) < \frac{\delta \operatorname{DIAM}(S_{U_0})}{2^k}$
- $\operatorname{DIAM}(S_{U_k}) < \frac{\operatorname{DIAM}(S_{U_0})}{2^k}$.

-17-

THUS (u_k) CAUCHY $\Rightarrow u_k \to V$.

MOREOVER $\mathrm{DIAM}(S_U) = 0 \Rightarrow U$ FIXED BY G_n.

$\ast\!\!\ast$

STEP 2 ω-A.S. $\exists\, y_n \in B(x_n, 2\delta_n \mathrm{DIAM}(S_{x_n}))$

S.T. $\forall\, z \in B\left(y_n, \dfrac{\delta_n \mathrm{DIAM}(S_{y_n})}{2}\right)$

$\mathrm{DIAM}(S_z) \geq \dfrac{\mathrm{DIAM}(S_{y_n})}{2}$.

TAKE $X_\infty =$

$$= \lim \left(X_n, \lambda_n d_n, y_n \right)$$

$$\lambda_n = \frac{2}{\mathrm{DIAM}(S_{y_n})} .$$

THE ACTIONS $G_n \curvearrowright X_n$ INDUCE AN ACTION $G_\infty \curvearrowright X_\infty$ S.T. $\forall\, z_\infty$,

$$\mathrm{DIAM}(S z_\infty) \geq 1.$$

QED

-19-

COR. (SHALOM'S THM) $\forall$ G F.G. WITH (T)

IS THE QUOTIENT OF A F.P. GP. WITH (T)

PROOF $G = \langle S \mid R_1, \dots, R_m, \dots \rangle$

$G_n = \langle S \mid R_1, \dots, R_n \rangle$

ASSUME $\forall n \in \mathbb{N}$, G_n ACTS ON A HILBERT SP. $\mathcal{H}_n$ WITHOUT GLOBAL FIXED POINT.

APPLY PREVIOUS $\Rightarrow$ $G_\omega = G$ ACTS ON

$\underbrace{\mathrm{LIM}_\omega(\mathcal{H}_n, \chi_n, \lambda_n)}_{\text{HILBERT SPACE}}$ WITHOUT FIXED POINT.

$\#$

QED

-20-

REM. SHALOM'S THM. WORKS WHEN WE REPLACE "FIXED POINT PROP. FOR HILB. SP." = (T) BY "F. P. P FOR $\mathcal{C}$ CLASS OF M.S. STABLE BY RESCALING AND ULTRALIMITS"

SECOND APPLICATION

TAKE G F.G. WITH (T)

$$\mathcal{P}(G) = \{ p \in (0, +\infty) \text{ s.t. } G \text{ HAS FL}^p \}$$

WE HAVE SEEN $(0, 2] \subseteq \mathcal{P}(G)$.

-21-

THM. (FISHER-MARGULIS)

$\mathcal{P}(G)$ IS OPEN.

PROOF WE PROVE $(0,+\infty) \setminus \mathcal{P}(G)$ CLOSED:

TAKE (p_n) IN $\nearrow$, $p_n \to p$

$\Rightarrow$ G ACTS ON SOME $L^{p_n}(X_n, \mu_n)$ WITHOUT F.P.

$\Rightarrow$ G ACTS ON $\underbrace{\lim_\omega L^{p_n}(\dots)}_{L^p(Y,\nu)}$ WITHOUT F.P. (THM)

$\Rightarrow p \notin \mathcal{P}(G)$.

QED

-22-

OTHER INCOMPATIBILITY RESULTS

PROP. IF G WITH (T) AND H THAT IS A-T-MENABLE, G, H F.G.

$\forall\, \varphi : G \to H$ HOMOM. HAS FINITE IMAGE.

COROLLARY IF G HAS (T) THEN G^{ab} IS FINITE.

PROOF OF COR. $G^{ab} = \underbrace{F}_{\text{finite}} \times \mathbb{Z}^n$.

THEN $\varphi : \underbrace{G}_{(T)} \to \underbrace{\mathbb{Z}^n}_{\text{a-T-M.}}$ ONTO.

$\boxed{\text{QED}}$

-23-

PROOF OF PROP.

TAKE $\varphi: G \to H$ HOM.

H A-T-MEN. $\Rightarrow \exists\ \theta: H \times H \to [0, +\infty)$

CND KERNEL LEFT INV. & PROPER:

$\forall\ M > 0,\ \exists\ D$ S.T.

$d(h, h') \geq D \Rightarrow \theta(h, h') \geq M.$

TAKE $\theta \circ \varphi \times \varphi : G \times G \to [0, +\infty)$

CND KERNEL ON G, LEFT-INV. $\Rightarrow$

$\Rightarrow \theta \circ \varphi \times \varphi$ BOUNDED BY $M_0 \Rightarrow$

GHAST

$$\Rightarrow d_H(\varphi(g), \varphi(g')) \leq D_0 = D_0(M_0)$$

$$\Rightarrow \operatorname{Im}\varphi \subseteq B_H(1, D_0).$$

QED

-24-

UNITARY DEF. OF (T)

TAKE UNITARY REP. ON H HILBERT H:

i.e. MOMOM. $G \longmapsto \underbrace{U(H)}_{\text{UNITARY OPERATORS}}$ S.T.

$\forall v \in H$, THE ORBIT MAP IS CONT.

$g \longmapsto gv$.

-25-

$\forall S \subseteq G$, $\forall \varepsilon > 0$, AN (S, ε)-ALMOST INV. VECTOR WRT π IS $v \in H$ S.T.

$$\sup_{s \in S} \|\pi(s)v - v\| \leq \varepsilon \|v\|, \quad v \neq 0.$$

. WE SAY π HAS ALMOST INVARIANT VECT. IF $\forall K \subseteq G$, K COMPACT, $\forall \varepsilon > 0$,

$\exists$ A (K, ε)-ALMOST INV. VECTOR.

$$G = \bigcup_{n \in \mathbb{N}} K_n$$

-26-

DEF. (KAZHDAN) G LOCALLY CPCT., 2-ND COUNTABLE S.T. EVERY UNITARY REP. ON A HILBERT WITH ALMOST INV. VECTORS HAS GLOBAL INV. VECTORS:

$$\exists\, v \neq 0 \text{ S.T. } \pi(g)v = v, \ \forall g \in G.$$

THEN G IS SAID TO HAVE PROP. (T).

-27-

THM. IF G HAS (T) THEN G IS COMPACTLY GEN.

PARTICULAR CASE IF G IS COUNTABLE WITH (T)
THEN G IS F.G.

PROOF TAKE $H \leq G$, H F.G.

$G \curvearrowright G/H$ TRANSITIVELY $\Rightarrow$

$\Rightarrow$ A UNITARY REP. π_H OF G ON $\ell^2(G/H)$.

TAKE THE ℓ^2-SUM

$$V = \bigoplus_{H \leq G \text{ f.g.}} \ell^2(G/H)$$

-28-

THE COMPLETION OF V, H, IS A HILBERT SP.

$(\pi_H)_H$ INDUCES A UNITARY REP. OF G ON H.

THE COMPACT SUBSETS OF $G = K$ FINITE.

THE REP. ABOVE HAS ALM. INV. V.

$\forall K$ FINITE, $\forall \varepsilon > 0$, $H = \langle K \rangle$.

K FIXES THE NON-ZERO VEC. 1_H IN $L^2(G/H)$

$\Downarrow (T)$

$\exists v \in H$ GLOBALLY FIXED.

$\Rightarrow \exists\, x^* \in H^*,\ x^* = \langle v, \cdot \rangle$

GLOBALLY FIXED BY G.

$x^* \neq 0 \Rightarrow x^*|_V \neq 0$ (V DENSE IN H)

$\Rightarrow x^*\Big|_{\ell^2(G/H)} \neq 0$ FOR SOME G/H

$\Rightarrow$ G HAS A GLOBAL FIXED POINT IN $\ell^2(G/H)$,

$f : G/H \to \mathbb{C}$. BUT G ACTS TRANSITIVELY $\Rightarrow$

$\Rightarrow f$ MUST BE CONSTANT $\Rightarrow$ G/H FINITE

QED