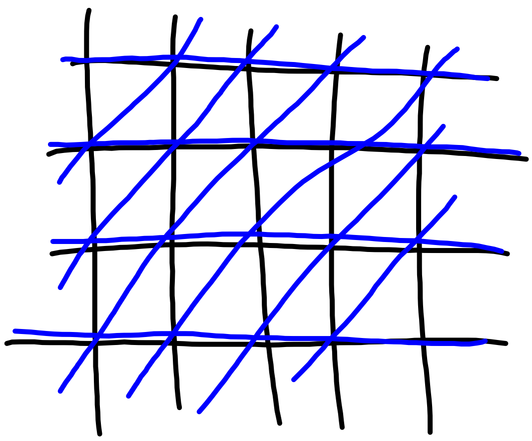
$$=2=$$

Ex. ① $\Gamma = \mathbb{Z}^2, \quad S = \{(\pm 1, 0), (0, \pm 1)\}$

$\mathrm{CAY}(\Gamma, S)$:



$S' = \{(\pm 1, 0), \pm(1,1)\}$

-4-

WE IDENTIFY $\mathbb{Z}^2 \to$ ORBIT

$(\mathbb{R}^2, \| \|_p)$, $1 < p < \infty$,

$$\begin{aligned} \mathbb{Z}^2 &\longrightarrow \mathbb{R}^2 \\ g &\longmapsto gx \end{aligned}, \quad \forall x \in \mathbb{R}^2$$

$\mathbb{Z}^2$ HAS $\Big\langle$ WORD METRICS / INDUCED FROM $\mathbb{R}^2$. EVEN BETTER, COMPARE $(\mathbb{Z}^2, d_S)$, $(\mathbb{R}^2, d)$.

=6=

A GENERALIZATION OF: $\mathbb{Z}^2$ quasi-isom. To $\mathbb{R}^2$:

THM. (MILNOR-SCHWARTZ)

LET (X, d) BE PROPER AND GEODESIC.
LET $\Gamma \curvearrowright X$ BY ISOMETRIES, PROPERLY DISC. S.T. $\Gamma \backslash X$ COMPACT. THEN Γ IS FINITELY GEN. AND (Γ, d_S), S AS ABOVE, IS Q.I. TO (X, d)

COROLLARY GIVEN M COMPACT RIEMANNIAN, $\tilde{M}$ UNIV. COVER IS Q.I. TO $\pi_1(M)$ WITH SOME d_S.

=10=

OTHER EX. OF QUASI-ISOM.:

1. BOUNDED PERTURBATIONS OF IDENTITY:

$$f : X \to X, \quad d(f(x), x) \leq C,$$

C UNIF. IN. $x \in X$

$\mathcal{B}(X) =$ ALL MAPS AS ABOVE.

= 12 =

A VIRTUAL ISOM. DEFINES A Q.I. (EX.)

IN MOST CASES, Q.I. ARE NOT IN $\mathcal{B}(X)$ OR V. ISOMORPHISMS:

EX. 1 $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1, x_2 + \delta(x_1))$

$\delta : \mathbb{R}_1 \to \mathbb{R}_1$ LIPSCHITZ (x_1, x_2)

=14=

TAKE $SOL = \mathbb{R}^2 \rtimes \mathbb{R}$

$$\mathbb{R} \longrightarrow Aut(\mathbb{R}^2)$$

$$t \longrightarrow \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

<u>EX.</u> VIA DIAGONALIZ. OF A, $\Gamma_A \hookrightarrow SOL$, AS A DISCRETE SUBGP., SOL/Γ_A COMPACT.

=16=

$$H \underset{fi}{\leqslant} \Gamma_A \Rightarrow H = \mathbb{Z}^2 \rtimes_{A^k} \mathbb{Z}$$

FOR SOME $k \in \mathbb{N}$, $k \geqslant 1$.

<u>EX.</u> $\mathbb{Z}^2 \rtimes_{A^k} \mathbb{Z} \simeq \mathbb{Z}^2 \rtimes_{B^m} \mathbb{Z} \Leftrightarrow A^k, B^m$ CONJUGATE

-18-

GIVEN F_2, MAP $F_2 \xrightarrow{\varphi_m} \mathbb{Z}_m$

$$a \longmapsto 1$$
$$0 \longrightarrow 0$$

$\ker \varphi_m \simeq F_{m+1}$.

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=20=

- $\mathrm{ISOM}(X)$ EMBEDS IN $QI(X)$

$$f \mapsto [f].$$

$$\mathcal{I}_X : \mathrm{ISOM}(X) \to QI(X).$$

REM. $\mathcal{I}_X$ COULD HAVE LARGE KERNEL:

$X = (\mathbb{R}^n, \| \|_2)$, $\ker \mathcal{I}_X$ = ALL TRANSLATIONS.

=22=

RIGIDITY QUESTIONS WE CONSIDER

Q.I. RIGIDITY

(1) WHAT ALGEBRAIC PROP. ARE PRESERVED BY Q.I.?

(2) FOR WHAT METRIC SP. (MANIFOLDS) (X, d), DOES Q.I. $(\Gamma, d_S) \sim (X, d)$ IMPLY A STRUCTURAL PROP. OF Γ?

=24=

SURVEY OF RESULTS

① ALGEBRAIC PROP. INVARIANT BY Q.I.

THM. (STALLINGS): Γ Q.I TO F_2 THEN Γ IS VIRTUALLY FREE.

Γ IS VIRTUALLY (*) = $\exists\ \Gamma_1 \leq \Gamma$, Γ_1 WITH (*).
FIN. INDEX

FOR Q.I. NEVER BETTER THEN "VIRTUALLY": Γ Q.I. $\Gamma \times$ FINITE

THM. (GROMOV) Γ Q.I. TO N NILPOTENT THEN Γ IS VIRTUALLY NILPOTENT.

BECAUSE: Γ V. NILP. $\Leftrightarrow$ $\# B(1, n) \leq n^a$, $a \in \mathbb{N}$, $a > 1$.

=26=

THE SAME IS TRUE FOR:

$X = \mathbb{H}^4_{\mathbb{C}}$ (CHOW)

$X = \mathbb{H}^4_{\mathbb{H}}$ (PANSU)

$\mathbb{H}^2_{Ca}$

THESE $+ \mathbb{H}^4_{\mathbb{R}}$ ARE THE SYMM. SPACES OF NON-COMP. TYPE OF RK. 1

$\mathbb{R} \hookrightarrow X$

③ EX. OF SMALL $QI(X)$: =28=

EX. 1 $X = \mathbb{H}^n_{\mathbb{R}}$, $QI(X)$ LARGE BY COMP. TO $ISOM(X)$

$X = \mathbb{H}^n_{\mathbb{C}}$

EX. 2 WHEN $X = \mathbb{H}^n_{\mathbb{H}}, \mathbb{H}^2_{Ca}$, $QI(X) = ISOM(X)$.

EX. 3 SAME FOR KLEINER-LEEB: $QI(X) = SL(n,\mathbb{R})$.

$X = SL(n,\mathbb{R})$ OR $X = \mathcal{S}_n$

REFORMULATION OF HYP.: $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ DISCRETE

$\mathbb{H}^n_{\mathbb{R}}/\Gamma$ FIN. VOL., NOT COMPACT.

TUKIA'S THM APPLIED TO Γ WITH COMPACT Q:

Λ Q.I. Γ $\Rightarrow$ Λ is VIRT. ISOM. $\Lambda_1 \leq \mathrm{Isom}$.

$(\Leftrightarrow$ QI. $\mathbb{H}^n_{\mathbb{R}})$ SAME PROP (DISCRETE, COMP Q.)

BUT Λ_1, Γ MIGHT NOT BE V.I.

BECAUSE: ALL $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ DISCRETE, COMP. Q., ARE Q.I.

COUNTABLY MANY EQ. CLASSES/VIRTUAL ISOM.

= 32 =

MOREOVER $QI(SL(n,\mathbb{Z})) \simeq SL(n,\mathbb{Q}) =$

$= \mathrm{COMM}_{SL(n,\mathbb{R})} SL(n,\mathbb{Z})$

HERE $\forall g \in SL(n,\mathbb{R})$ ACTS ON $SL(n,\mathbb{Z})$ BY

$x \longmapsto gxg^{-1}$

DEF. $\Gamma \leq G$, Γ f.g. $\mathrm{COMM}_G \Gamma = \{ g \in G ;\ g\Gamma g^{-1} \cap \Gamma$ is f.i. in Γ, $g\Gamma g^{-1} \}$

=34=

(2) EX. (BURGER-MOZES) OF $\Gamma \curvearrowright T_n \times T_m$

Γ BY ISOM., P.D. WITH $T_n \times T_m / \Gamma$ COMP. (PROPERLY DISC.)

S.T. Γ SIMPLE (NO NON-TRIVIAL $N \triangleleft \Gamma$).

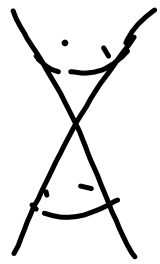
NOTE Γ Q.I. $F_n \times F_m$.

= 36 =

GROMOV - PIATETSKII SHAPIRO

$q : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ OF SIGNATURE $(n, 1)$

$q(x) = a_1 x_1^2 - a_2 x_2^2 - \dots - a_n x_{n+1}^2$. $a_i \in \mathbb{Z}$



$SO(q) \simeq \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$

$\Gamma = SO(q) \cap SL(n, \mathbb{Z})$

Γ unif. $\Leftrightarrow \{ v \in \mathbb{Z}^{n+1}, q(v) = 0 \} = \{ 0 \}$, $n \leq 4$.