THE RIGIDITY RESULTS OF KLEINER-LEEB

THM. LET X BE AN IRREDUCIBLE

- NON-FLAT SYMMETRIC SPACE OF NON-COMP. TYPE

OR

- EUCLIDEAN BUILDING THAT IS THICK, WITH COCOMACT AFFINE WEYL GROUP.

ASSUME $\operatorname{RANK} X \geq 2$.

THEN $\forall\, q : X \to X$ (L, C)-QUASI-ISOMETRY, $L \geq 1, C \geq 0$, $\exists$ A SIMILARITY $\sigma : X \to X$ S.T.

$$d(q(x), \sigma(x)) \leq D, \quad D = D(L, C).$$

=2=

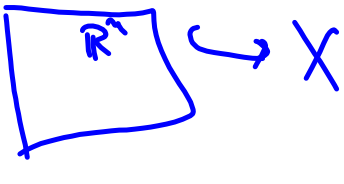
SIMILARITY: $\sigma: X \to X$, $d(\sigma(x), \sigma(y)) =$

$= \lambda\, d(x,y)$, $\forall x, y$, $\lambda \in (0,\infty)$ FIXED,

σ BIJECTION.

X IRRED. $=$ X NOT $\neq$ PROD. $X_1 \times X_2$

RANK X = MAX $\{ m ;$ $\exists$ AN ISOM. EMBEDDING $(\mathbb{R}^m, d_{\|\cdot\|_2}) \hookrightarrow X$



-3-

GOAL IN WHAT FOLLOWS = DEFINE ALL + PROOF.

ROUGH IDEA OF MEANING OF "EUCL. BUILDING"

· INTROD. BY J. TITS TO UNDERSTAND ALGEBRAIC GROUPS.

REF.: J. TITS - BUILDINGS OF SPHERICAL TYPE AND FINITE BN-PAIRS, LN 386, SV, 1974

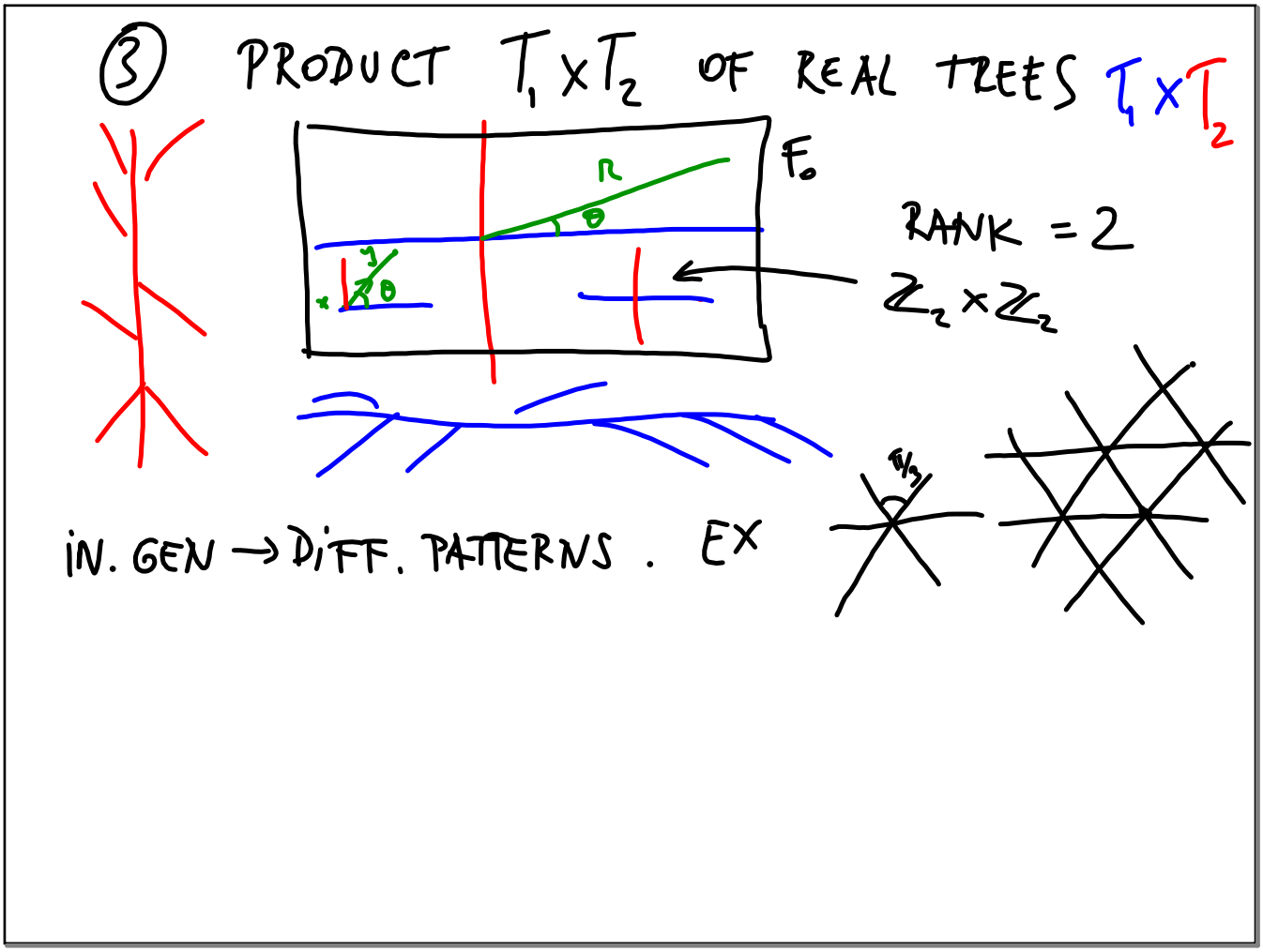
DEF IS ABSTRACT. HERE: A METRIC APPROACH

AS FOR SIMPLICIAL CPLX ⟶ GEOM. REALIZ.

=4=

EX. ① SIMPLICIAL TREE = GRAPH WITHOUT CYCLES (REGULAR)

② REAL TREES = METRIC SP. (X, d) (COMPLETE) S.T. $\forall x, y \in X$ ARE JOINED BY A UNIQUE ARC

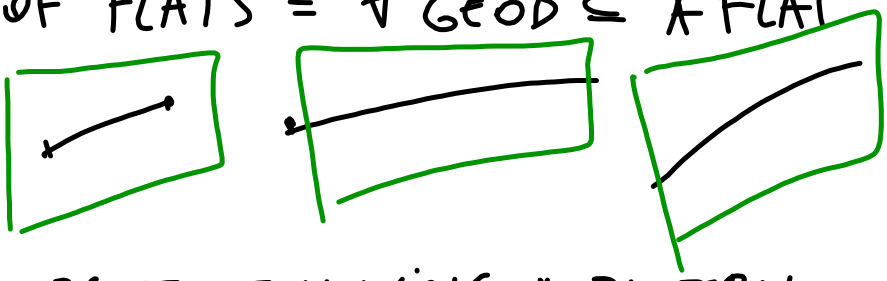

③ PRODUCT $T_1 \times T_2$ OF REAL TREES $T_1 \times T_2$
F_0
R
θ
RANK = 2
$\mathbb{Z}_2 \times \mathbb{Z}_2$
IN. GEN → DIFF. PATTERNS . EX
π/3

=6=

MAIN FEATURES:

- A COMBINATION OF "FLAT" AND "TRANSVERSAL GEOM. OF NEGATIVE CURVATURE".

. PLENTY OF FLATS = $\forall$ GEOD $\subseteq$ A FLAT



. FLATS INTERSECT FOLLOWING A PATTERN.

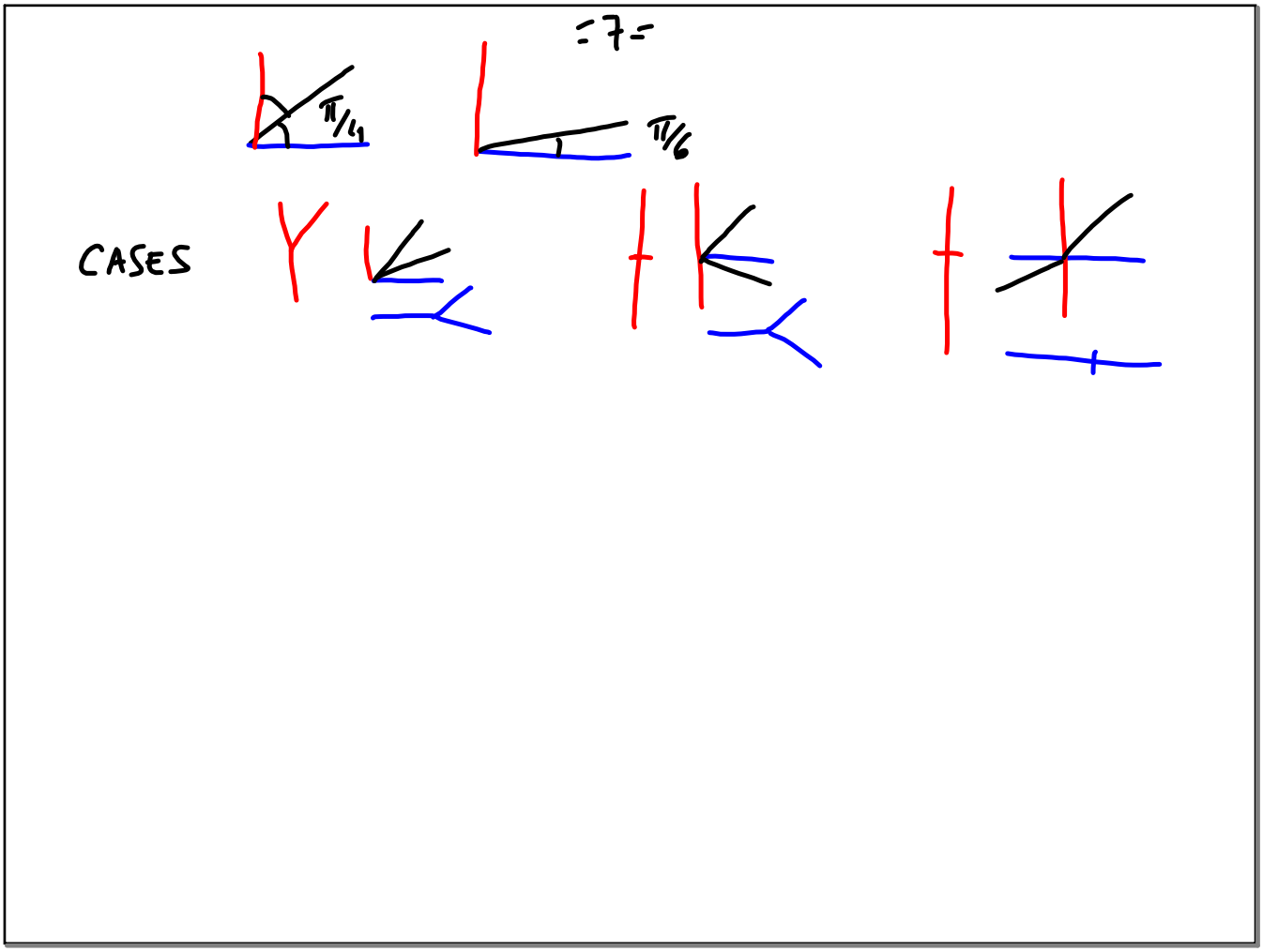
FIX F_0, LOOK AT $F_0 \cap F$:

IN PART. THE LINES OF FORM $F_0 \cap F \to$? DIR.

. IN A POINT THEY DETERMINE A FIXED GROUP OF REFL.

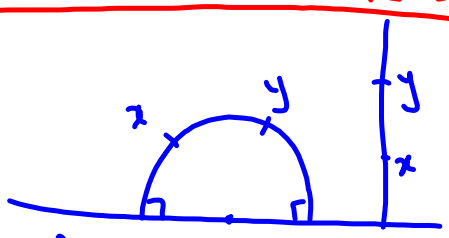
. EACH RAY & ORIENTED GEODESIC COMES WITH A SLOPE.

. TWO RAYS / ORIENTED GEOD. CAN FORM ANGLES TAKING FINITELY MANY VALUES, DEPENDING ON THEIR SLOPES


=7=
π/4
π/6
CASES

ROUGH IDEA ABOUT SYMMETRIC SP.

EX. ① $\mathbb{H}^n_{\mathbb{R}}$ $\mathbb{H}^2_{\mathbb{R}}$

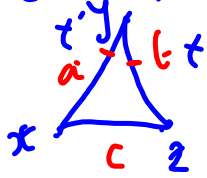


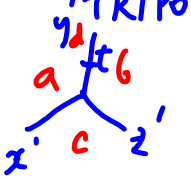
RANK = 1

② $\mathbb{H}^n_{\mathbb{R}} \times \mathbb{H}^m_{\mathbb{R}}$

RANK 2

$\mathbb{H}^n_{\mathbb{R}}$ IS GROMOV $\mathbb{R}$-x/o HYPERBOLIC: $\forall$ GEOD. TRIANGLE, GIVEN THE 'TRIPOD':





GEOD Δ = BDED PERTURB. OF TRIPODS $d(t', t'') \leq \ln 2$

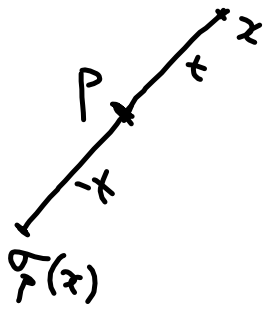
$\mathbb{H}^n_{\mathbb{R}} \approx$ "THICKENING"/BDED PERTURB. OF A TREE.

=9=

SYMM. SPACES ARE "RIEMANNIAN THICKENINGS" OF EUCLIDEAN BUILDINGS".

REF. S. HELGASON'S BOOKS

· SYMMETRIC SPACE : A SIMPLY CONNECTED RIEMANNIAN MANIFOLD X S.T. $\forall\, p \in X$, $\exists$ A GLOBAL ISOMETRY DEF. AS A GEOD. SYMMETRY:



· $G = \mathrm{ISOM}_{\mathrm{Identity}}(X)$ ACTS TRANSITIVELY ON X.

=10=

• G IS A SEMISIMPLE CONNECTED LIE GROUP.

CONVERSELY: GIVEN G AS ABOVE, K A MAXIMAL COMPACT SUBGP., G/K IS A SYMMETRIC SPACE.

. $\forall G \rightsquigarrow \mathfrak{g}$ ITS LIE ALGEBRA (SEMISIMPLE)

$\exp: \mathfrak{g} \to G$, $\log: G \to \mathfrak{g}$.

=11=

THUS WE HAVE A "STACK" OF 3 THEORIES, AND A TRIPLE DICTIONARY:

$\mathfrak{g}$ ↓ SSPLE LIE ALGEBRA

$G = \mathrm{ISOM}_{Id}(X)$ SSPLE LIE GP.

↓

X RIEMANNIAN GEOM.

=12=

CAN ALSO USE ALGEBRAIC GEOMETRY:

EX. OF SEMISIMLE LIE GPS: $SL(n,\mathbb{R})$, $SO(p,q)$

- EVERY CONN. SEMISIMPLE LIE GP. APPEARS AS $\overline{G}_0$, $\overline{G}$ ALGEBRAIC GROUP, $\overline{G}_0$ ITS ID. COMP.
- $\exists$ AN EMBEDDING $\varphi: \overline{G} \longrightarrow SL(n,\mathbb{R})$ S.T. $\varphi(\overline{G})$ IS A REAL ALG. SUBGP., $\varphi(\overline{G})^T = \varphi(\overline{G})$.

IF $\overline{K}$ IS A MAX. CPT. SUBGP., $\varphi(\overline{K}) = \varphi(\overline{G}) \cap SO(n,\mathbb{R})$

ONISCHIK-VINBERG – LIE GPS & ALGEBRAIC GROUPS SV '90.

=13=

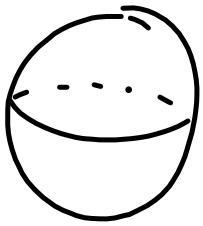
SPHERICAL AND EUCLIDEAN BUILDINGS

METRIC NOTION OF CURVATURE

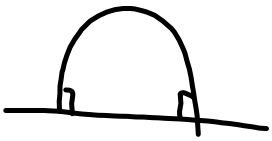
GIVEN $k \in \mathbb{R}$, CONSIDER M^2_k THE 2-DIMENS. MODEL SPACE FOR CURVATURE k.

$k=0$: $\mathbb{R}^2 + d_{\|\|_2}$

$k=1$: $S^2 + d_{S^2}$



$k=-1$: $\mathbb{H}^2_{\mathbb{R}}$



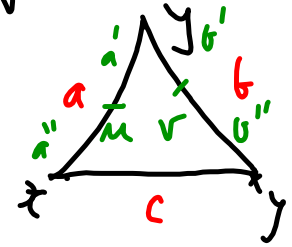
=14=

$D(k)$ = DIAM. OF M^2_k $\begin{cases} \infty \text{ IF } k \leq 0, \\ < \infty \text{ IF } k > 0. \end{cases}$

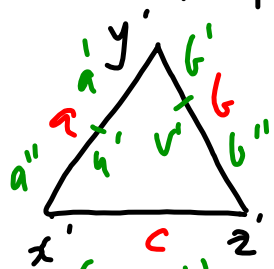
DEF. A METRIC SP. (X, d) IS CAT(k) ("METRICALLY OF CURVATURE $\leq k$") IF:

(1) $\forall\, x, y \in X$, $d(x,y) < D(k)$, $\exists$ A UNIQUE GEODESIC $x \rightsquigarrow y \leftarrow$ ISOM. EMB. OF $[0, d(x,y)]$.

(2) $\forall$ GEODESIC TRIANGLE WITH PERIMETER $< 2D(k)$ IN (X,d):



GIVEN A Δ IN M^2_k:



$d_X(u,v) \leq d_{M^2_k}(u', v')$

(3) (X,d) COMPLETE

N.B. X NOT LOCALLY COMPACT!

=15=

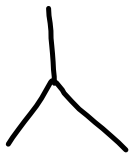
EXAMPLES (1) X SIMPLY CONNECTED, COMPLETE RIEMANNIAN MAN. WITH SECTIONAL CURVATURE $\leq k \Rightarrow$ CAT(k). (GROMOV)

CAT = CARTAN - ALEKANDROV - TOPOGONOV

CHAT = HADAMARD

(2) REAL TREES ARE CAT($-k$), $\forall k>0$

M^2_k CAT($-\infty$)



=16=

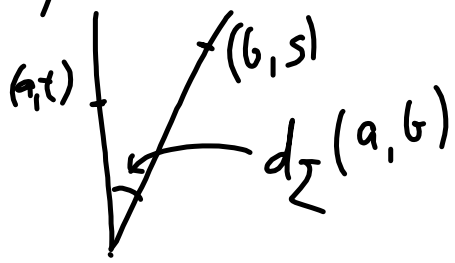
METRIC CONES

TAKE (Σ, d), $\mathrm{DIAM}(\Sigma) \leq \pi$.

$$\mathrm{CONE}(\Sigma, d) = \Sigma \times [0, +\infty) \Big/ \Sigma \times \{0\}$$

+ METRIC: $\forall\ (a,t), (b,s)$:

CONSIDER IN $\mathbb{R}^2$



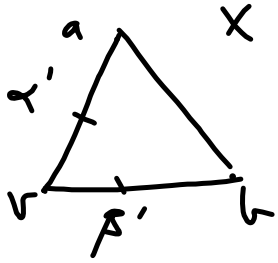
=17=

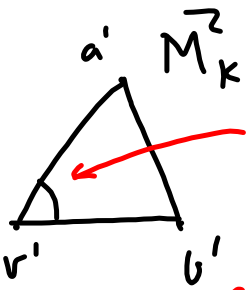
PROP. IF (Σ, d) IS CAT(1) THEN CONE (Σ, d) IS CAT(0).

ANGLES AND SPACE OF DIR.

WE WANT SOME "METRIC VERSION" OF THE TANGENT SPACE + ANGLES:

$\forall\, r, q, b$:





DENOTED $\tilde{\angle}_r(q, b)$,

CALLED COMPARISON ANGLE.

EX. $\forall\ \alpha' \in (v, a]\ ,\ \beta' \in (v, b]\ ,\ \tilde{\angle}_v(\alpha', \beta') \leq \tilde{\angle}_s(a, b)$

THIS IMPLIES $\exists \lim_{\substack{\alpha' \to v \\ \beta' \to v}} \tilde{\angle}_s(\alpha', \beta')$

LEMMA THE ABOVE $= \lim_{\alpha' \to v} \tilde{\angle}_v(\alpha', b)$.

DEF. THE ANGLE $\angle_v(a, b)$ = LIMITS ABOVE.

Fix $v \in X$.

THE SPACE OF DIR. IN v, $\Sigma_v^* X =$

$$\left\{ [v,a] \text{ OR } [v,a) \text{ ; INFINITE OR FINITE} \right\} \Big/ \sim$$

$[v,a] \sim [v,b] \iff \angle_v(a,b) = 0.$

- CONSIDER THE DISTANCE DEFINED BY ANGLE.
- TAKE THE COMPLETION $(\Sigma_v X, d)$.
- TAKE THE METRIC CONE $\text{CONE}(\ldots) =: (C_v X, d)$.

IDEA: $\Sigma_v X$ is "THE UNIT TANG. SP."

$C_v X$ is "THE TANGENT SP."

LEMMA IF X is CAT(1), $\Sigma_v X$ is CAT(1), $\forall v$,

$C_v X$ is CAT(0).

=20=

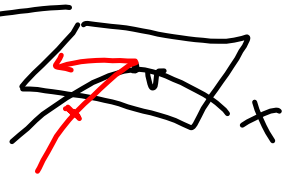
LEMMA. $\exists$ A LOGARITHM MAP:

$$\text{LOG}: B(v, D(k)) \to C_v X$$

$$\text{LOG}(x) \longrightarrow \left(\overrightarrow{[v,x)}, d(v,x)\right)$$

$$\text{LOG}(v) \longrightarrow \text{COLLAPSED}\left(\Sigma_v X\{0\}\right)$$

REM. IN GEN., NO EXP. MAP: T_x



-21-

CAT(0) - SPACES

(X, d) CAT(0) SPACES $\longrightarrow$ $\partial_\infty X$ BOUNDARY AT ∞, CAT(1)