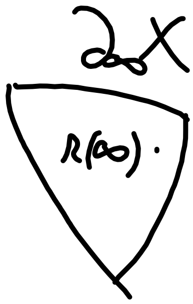
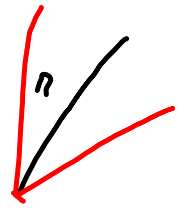
$\boxed{Q}$ Take r_1, r_2 GEOD RAYS.

$r_1 \subset F_1 \quad r_2 \subset F_2$

GIVEN $g \in G$ S.T. $g r_1 = r_2$, CAN ONE CHOOSE g S.T. $g F_1 = F_2$.

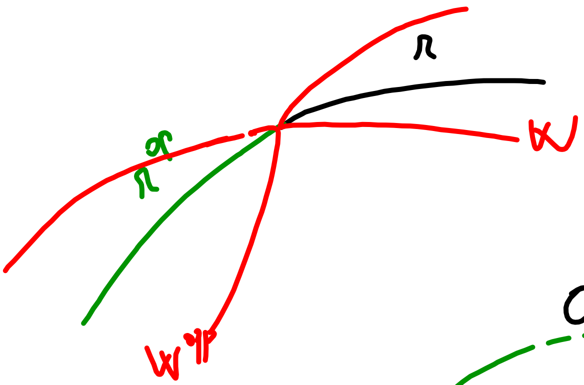
<u>REMARK</u> X SYMM. SPACE

(1) IF r HAS REGULAR SLOPE $\Leftrightarrow$ 

$\Leftrightarrow$ 

THEN $\exists!$ F CONTAINING r.

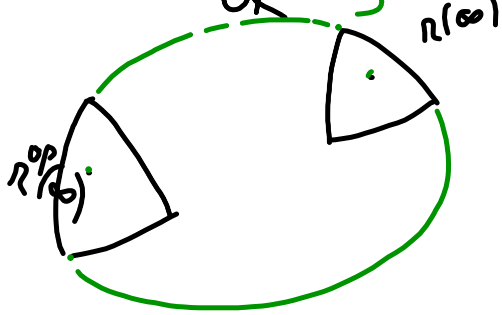
=2=



$\mathrm{Hull}(W, W^{opp}) = \mathrm{Flat}$

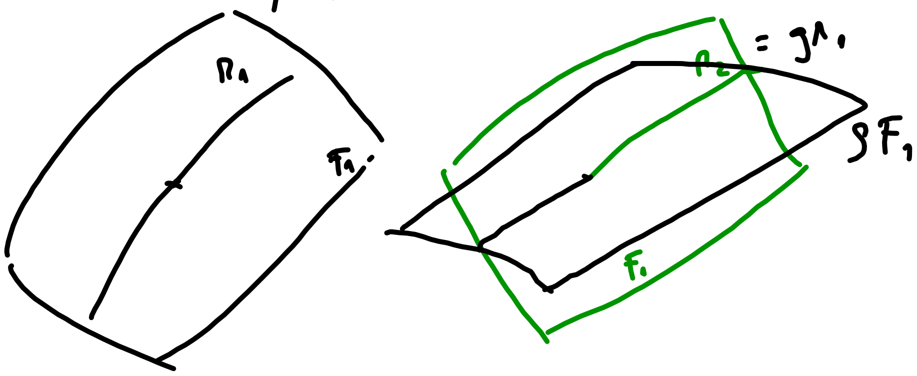
OR

S^{n-1} $(\partial_\infty X, \angle_{Tits})$



-3-

IF r_1, r_2 SINGULAR THEN WE MIGHT HAVE



G ACTS TRANS. ON (FLAT, BASEPT) $\Leftrightarrow$

$\Leftrightarrow$ $K = \mathrm{STAB}(x_0)$ —||— (FLAT, x_0).

=4=

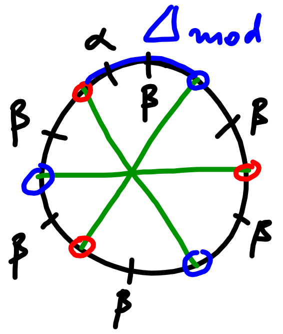
EX. DEDUCE FROM THE ABOVE THAT, GIVEN ϕ SINGULAR FLAT ($\phi \simeq \mathbb{R}^k$, $k < n$, $\phi = F_1 \cap .. \cap F_{n-k}$)

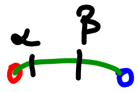
$K = STAB(\phi)$ ACTS TRANSITIVELY ON $\{F$ CONTAINING $\phi\}$.

-5-

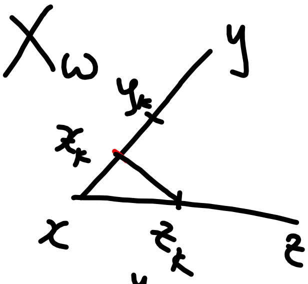
EX. TAKE X E.B., CORRESP. FINITE COXETER CPLX

IS





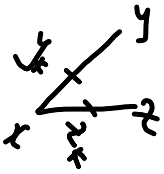
$D(\alpha,\beta)$



NOTATION $\theta(\overline{xy}) = \theta_1$

$\theta(\overline{xz}) = \theta_2$, $\theta(\overline{yz}) = \theta_1^{op}$

$\angle_x(y,z) = \alpha$



$$\angle_x(y,z) = \lim_{y_k\to x,\, z_k\to x} \tilde{\angle}_x(y_k, z_k) = \lim_{\substack{x_k \to x \\ \cdots}} \tilde{\angle}_{x_k}(y_k, z_k)$$

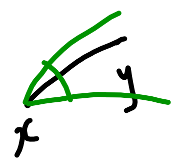
=6=

THEOREM EVERY AS. CONE OF A SYMMETRIC SPACE IS A EUCLIDEAN BUILDING MODELLED ON $(\mathbb{R}^n, W_{\text{aff}})$.

END OF PROOF (CORRECTION)

(EB2) $x, y, z \in X_\omega$ AS. CONE, x, y, z PAIRWISE DISTINCT, $\angle_x(y,z) \in D(\theta(\overline{xy}), \theta(\overline{xz}))$.

$\theta : \{\text{GEOD SEGM., RAYS}\} \to \Delta_{mod}$



-7-

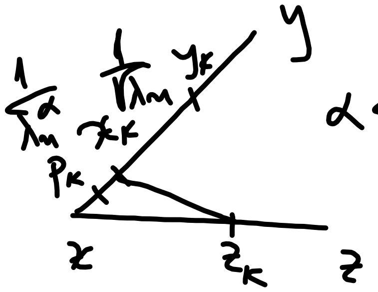
$$\forall k, \quad \left| \tilde{\angle}_{x_k}(y_k, z_k) - \alpha \right| < \varepsilon_k$$

$$\left| \tilde{\angle}_{x_k}(p_k, z_k) - (\pi - \alpha) \right| < \varepsilon_k$$

$\forall k$, PICK n_k S.T. $\tilde{\angle}_{x_{k,n_k}}\left(y_{k,n_k}, z_{k,n_k}\right)$ IS ε_k-CLOSE FROM $\tilde{\angle}_{x_k}(y_k, z_k)$. LIKEWISE IIND ANGLE

= 8 =

MOREOVER, FOR A GOOD CHOICE OF $p_k \to x$,



$\alpha < 1$ $\quad \tilde{\angle}_{x_k}(p_k, z_k) \to \pi - \alpha$

IN X_ω, $a_\omega = [a_n]$, IN PART.

$x_k = [x_{k,n}], \ldots$

$X_\omega = \mathrm{CONE}(X, x_n, \lambda_n)$, $\lambda_n \to \infty$

FOR FIXED a, b, c, $\tilde{\angle}_{a_n}(b_n, c_n) \to \tilde{\angle}_a(b, c)$

= 9 =

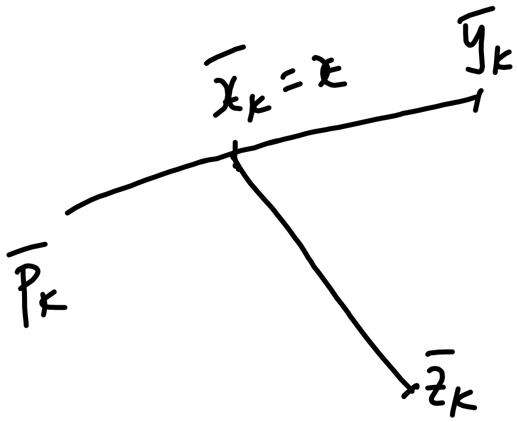
IN THE END WE OBTAIN SEQ $\bar{x}_k = (x_k, u_k)$

$\bar{y}_k = (y_k, u_k)$ .... S.T.

$$\tilde{\angle}_{\bar{x}_k}(\bar{y}_k, \bar{z}_k) \to \alpha$$

$$\tilde{\angle}_{\bar{x}_k}(\bar{p}_k, \bar{z}_k) \to \pi - \alpha$$

WHILE $d(\bar{x}_k, \bar{y}_k)$, ETC. $\to +\infty$.



$=10=$

$$\lim \theta(\overline{\overline{x}_k\,\overline{y}_k}) = \lim(\overline{x_k y_k}) = \theta_1$$

$$\lim \theta(\overline{\overline{x}_k\,\overline{p}_k}) = \theta_1^{op}$$

$$\lim \theta(\overline{\overline{x}_k\,\overline{z}_k}) = \theta_2$$

WLOG, APPLY ISOM., ASSUME $\overline{x}_k = x$ FIXED PT.

$[x, \overline{y}_k] \to [x, a)$ OF SLOPE θ_1

$[x, \overline{p}_k] \to [x, b)$ OF SLOPE θ_1^{op},

$[x, \overline{z}_k] \to [x, c)$ SLOPE θ_2

=11=

<u>RECALL</u> $\forall a,b \in \partial_\infty X$, $x_k \to a$, $x_k \in X$
$y_k \to b$, $y_k \in X$,

AND c_0 FIXED POINT IN X

$$\lim \tilde{\angle}_{c_0}(x_k, y_k) \geq \angle_{Tits}(a,b)$$

THUS

$$\alpha = \lim \tilde{\angle}_x(\bar{y}_k, \bar{z}_k) \geq \angle_{Tits}(a,c)$$

$$\pi - \alpha = \lim \tilde{\angle}_x(\bar{z}_k, \bar{p}_k) \geq \angle_{Tits}(b,c)$$

$$\angle_{Tits}(a,b) + \angle_{Tits}(b,c) \geq \angle_{Tits}(a,b) = \pi.$$

=12=

THEN ALL INEQ. ARE EQ, IN PART.

$$\alpha = \angle_{Tits}(a, b) \in \mathcal{D}(\theta_1, \theta_2).$$

$\boxed{QED}$

RIGIDITY IN EUCL. BUILDINGS

THEOREM CONSIDER X IRREDUCIBLE EUCL. BUILDING. EVERY HOMEOM. $f: X \to X$ IS A SIMILARITY:

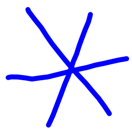
$$d(f(x), f(y)) = \lambda\, d(x,y), \quad \lambda > 0.$$

=13=

CONVENTIONS ALL E.B. CONSIDERED HAVE 3 PROP.:

- W_{aff} ACTS TRANSITIVELY ON $E \simeq \mathbb{R}^n \simeq F$.
  $W_{aff} = W \rtimes \mathbb{R}^n \subseteq \mathrm{ISOM}(\mathbb{R}^n)$
- THEY ARE THICK: $\forall$ SINGULAR HYPERPLANE IS $\subseteq$ AT LEAST 3 DISTINCT HALF APART.
- MOUFANG PROP.: STAB(HALF-APART) ACTS TRANS. ON THE HALF APARTS SHARING BRY WITH IT.

$\mathbb{R}^2$



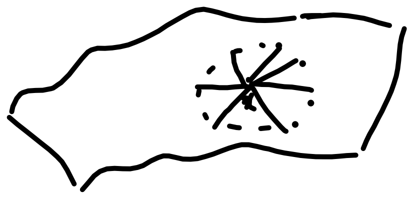
=14=

KEY STEP: KEY PROP. $\forall f : X \to X$, X E.B., f HOMEOM, f (APART.) = APART.

SKETCH OF ARGUMENT

KEY LEMMA $\forall f, X$ AS ABOVE, $\forall F$ APART., $f(F)$ HAS THE FOLLOWING LOCAL PROP.: $\forall x \in f(F)$, $\exists \varepsilon_x$ S.T. $f(F) \cap B(x, \varepsilon_x) = \bigcup_{i=1}^{N} W_i \cap B(x, \varepsilon_x)$, WHERE $W_1, \dots, W_N$ WEYL CH. WITH VERTEX x, AND IN $\Sigma_x X$, $\bigcup_{i=1}^{N} W_{i,x}$ HOMEOM. TO S^{R-1}!

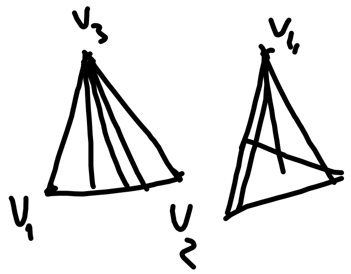
$f(F)$ =15=



IDEA OF PROOF

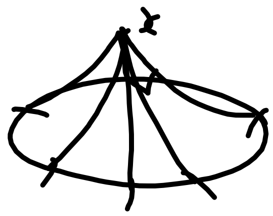
THURSTON : STRAIGHTENING OF SIMPLICES:

$\forall$ SINGULAR SIMPLEX $\sigma : \Delta_k \to X$, $STR(\sigma)$ DEFINED AS FOLLOWS. TAKE VERTICES WITH ORDER $\{v_1, v_2, \dots, v_{k+1}\}$.



NOTE: $STR(\sigma) \subseteq$ CONVEX HULL (IMAGE(σ))

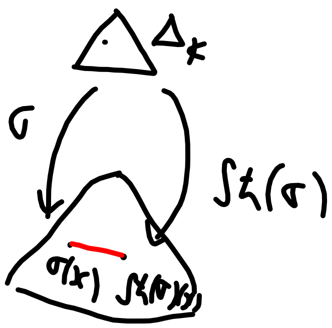
RECALL CONE(v, FLAT) = $\bigcup$ WEYL CH. WITH VERTEX x, BDRY $\subseteq \partial_\infty F$



CONSEQ $STR(\sigma) \subseteq \bigcup_{i=1}^{N}$ WEYL CH.

=16=

USE THE GEODESIC HOMOTOPY TO CONSTRUCT A CHAIN HOMOTOPY H BETWEEN THE CHAIN MAP STRAIGHTENING AND THE CHAIN MAP ID.



MOREOVER $H(\sigma)$ STAYS IN CONVEX HULL (IMAGE OF σ).

=17=

LEMMA $\forall\, V \subseteq U \subseteq X$, V, U OPEN,

$\forall\, k > n$, $H_n(U, V) = 0$.

PROOF TAKE $[c] \in H_n(U, V)$.

UP TO BARYC. SUBDIV., ASSUME CONVEX HULL $(\sigma) \subseteq U$

CONVEX HULL $(\partial\sigma) \subseteq V$.

THEN REPLACE EACH σ BY $STR(\sigma)$.

WE END UP WITH $[c] \subseteq \bigcup_{i=1}^{n} F_i$

$\boxed{\text{QED}}$

= 18 =

ALGEBRAIC TOP. ARG. IN SAME SPIRIT (SECTION § 6.2 IN KLEINER-LEEB) GIVE

KEY LEMMA $\forall$ TOPOLOGICAL EMBEDDING

$\varphi : B \to X$, $B = B(x, R) \subseteq \mathbb{E}^n \simeq \mathbb{R}^n$,

$\forall\, x \in B \setminus \partial B$, $\exists\, \varepsilon_x$ S.T. ....

-19-

PROOF OF KEY PROP. TAKE $f: X \to X$, $x \in X$, $y = f(x)$.

IN $\Sigma_x X$, TAKE AN ARBITRARY CHAMBER W_x, GERM OF WEYL CH.

$\exists$ F, F' APARTS. S.T. $W_x = F_x \cap F'_x$.

$$f(W)_y = \underbrace{f(F)_y}_{} \cap \underbrace{f(F')_y}_{} = \bigcup_{j=1}^{N} W'_j .$$

CONES OVER TOP. SP.

=20=

WE CANNOT HAVE $N \geq 2$.

OTHERWISE APPLY THE ARG. FOR f^{-1}:

$$W_x = \bigsqcup_{j=1}^{N} \underbrace{f^{-1}(W_j')} \quad \#$$

THEN $N = 1$, $f(W_x) = W_y'$. OBVIOUSLY IF $W_x \cap \tilde{W}_x$ IS A FACE OF DIM., SO IS $f(W_x) \cap f(\tilde{W}_x)$.

-21-

THUS, $\forall x$, f INDUCES AN ISOMETRY $\Sigma_x X \to \Sigma_y X$.

IN PARTICULAR, KEY LEMMA IMPROVED:

$\forall F$ FLAT / APT., $\forall y \in f(F)$, $\exists \varepsilon_y$ S.T.

$f(F) \cap B(y, \varepsilon_y) = F' \cap B(y, \varepsilon_y)$.

THEN $f(F)$ IS A SIMPLY CONN. SP. LOCALLY ISOM. TO $\mathbb{R}^2 \Rightarrow$

$\Rightarrow f(F) \simeq \mathbb{R}^2$.

$\boxed{\text{QED}}$

=22=

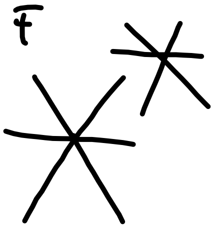
PROOF OF THEOREM $f : X \to X$ HOMEOM.

TAKE F APART. $\subseteq X$, $f(F) = F'$ APART.

$f : F \to F'$

F, F' COME WITH A STR. OF AFFINE COXETER CPLX

F



F COMES WITH 3 PARALLEL FOLIATIONS OF HYPERPLANES. THESE ARE THE ONLY HYP. IN F THAT CAN BE WRITTEN $F \cap F_n$.

= 23 =

THEN WE HAVE $n+1$ FOLIATIONS (BECAUSE X IRRED)

f(FOLIATION) = FOLIATION.

EX. INTRODUCE AFFINE COORD. S.T. THE FOLIATIONS ARE GIVEN BY $x_i = c$ OR $x_1 + \ldots + x_n = c$

S.T. $f(\{x_i = c\}) = \{x_i' = c\}$

=24=

PROVE . $f(mx) = m f(x)$, $m \in \mathbb{Z}$,

. $f(\lambda x) = \lambda f(x)$, $\lambda \in \mathbb{Q}$,

$\lambda \in \mathbb{R}$

. f LINEAR

. f IS A HOMOTHETY

$f(x) = \lambda_0 x$.

WE FOUND $f|_{FLAT}$ = HOMOTHETY.

IF $F \cap F' \neq \emptyset$, $f|_F$, $f|_{F'}$ HAVE SAME λ_0.

$\forall F, F', \exists F_1 = F, F_2, F_3, \dots, F_k = F'$ S.T. $F_i \cap F_{i+1} \neq \emptyset \Rightarrow$ SAME λ_0.

$\boxed{\text{QED}}$