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QUASI-ISOMETRIC CLASSIFICATION OF NON-UNIFORM LATTICES IN SEMISIMPLE GROUPS OF HIGHER RANK

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Abstract

We give another proof of the quasi-isometric classification theorem of non-uniform lattices in higher rank semisimple groups. We use the asymptotic cone and a class of flats (the logarithmic flats) which move away in the cusp with logarithmic speed.

1 Introduction

In this paper we consider finitely generated groups $\Lambda = \langle S \rangle$ endowed with some word metric [G2, §0]. The word metric depends on the chosen generating set. But two word metrics are quasi-isometric (Definition 3.3.1). It is natural then to study only those properties of Λ with some word metric which are invariant up to quasi-isometry. Among finitely generated groups we study non-uniform lattices in higher rank semisimple groups. A lattice in a semisimple group G is a discrete subgroup Γ such that $\Gamma\backslash G$ admits a finite G-invariant measure. If $\Gamma\backslash G$ is compact the lattice is called uniform, otherwise it is called non-uniform. We give another proof of the fact that the class of non-uniform lattices is "closed" with respect to quasi-isometric equivalence. More precisely, we prove the following result due to Alex Eskin [E]:

Theorem 1.1. *Let Λ be a finitely generated group and G a semisimple group of rank at least 2, with finite center and without factors of rank ≤ 1.*

(1) *If Λ is quasi-isometric to a non-uniform irreducible lattice Γ in G, then there exists an exact sequence*
$$0 \to F \to \Lambda \to \Gamma_1 \to 0\,,$$
where F is a finite group and Γ_1 a non-uniform lattice in G.

(2) *Let Γ and Γ_1 be two non-uniform lattices in G, at least one of which is irreducible. The lattices Γ and Γ_1 are quasi-isometric if and only if they are commensurable (i.e. there exists g in G so that $g\Gamma_1 g^{-1} \cap \Gamma$ has finite index in $g\Gamma_1 g^{-1}$ as well as in Γ).*

The previous theorem is a consequence of the following rigidity result.

Theorem 1.2 (rigidity). *Let G be a semisimple group of rank at least 2 with finite center and without factors of rank ≤ 1. Let Γ be a non-uniform irreducible lattice in G. If $q : \Gamma \to \Gamma$ is an (L, c)-quasi-isometry, then there exists a positive constant D depending only on L and c and an isometry g in the commensurator of Γ such that*

$$d\big(q(\gamma), g(\gamma)\big) \leq D\,, \quad \forall \gamma \in \Gamma\,.$$

We recall that the commensurator of a discrete subgroup Γ in a Lie group G is the subgroup $Comm(\Gamma)$ of G containing those elements g such that $g\Gamma g^{-1} \cap \Gamma$ is a finite index subgroup in $g\Gamma g^{-1}$ and in Γ.

The first results on quasi-isometric classification of non-uniform lattices have been obtained by Richard Schwartz. In [S1], he proved Theorems 1.1 and 1.2 for lattices in a semisimple group G of rank 1, different from $SL(2, \mathbb{R})$. He also obtained, together with Benson Farb, the quasi-isometric classification of Hilbert modular groups ([FS],[S2]). In [S2] he conjectured Theorems 1.1 and 1.2 in the higher rank case, which were proved by A. Eskin in [E]. We refer to [Mo] for a more complete history of the problem of classification of lattices up to quasi-isometry.

Our proof of Theorem 1.2 goes as follows. Let Γ be a non-uniform irreducible lattice in a semisimple group G of rank at least 2, and let X be the symmetric space associated to G. By removing open horoballs from X one can obtain a subspace X_0 in X on which Γ acts cocompactly (see for instance [L, Section 5]). This implies that Γ with a word metric is quasi-isometric to X_0 with its length metric. By the Lubotzky-Mozes-Raghunathan theorem ([LuMR1,2]), the length metric on X_0 is bilipschitz equivalent to the induced metric. So in our study of Γ up to quasi-isometry, we can always replace it by X_0 with the induced metric. We can define a Γ-invariant projection $\pi_0 : X \to X_0$.

We consider an (L, c)-quasi-isometry $q : \Gamma \to \Gamma$. We can extend it to an (L, c')-quasi-isometry $q : X_0 \to X_0$, $c' > c$. We want to use Tits' theorem ([T], see Theorem 2.3.2 in this paper) to find an isometry g at a finite distance from q. For this we need to add the hypothesis that G has no rank one factors. By means of q, we construct a simplicial isomorphism $\Phi : \partial_\infty X \to \partial_\infty X$, where $\partial_\infty X$ is the boundary at infinity of the symmetric space X with the canonical spherical building structure (section 2.3.A). There exist Weyl chambers in X moving away of X_0 very fast (e.g. Weyl chambers containing minimizing geodesic rays which project on minimizing rays in the cusps of $\Gamma \backslash X$). Therefore, the quasi-isometry q cannot induce from the beginning a bijection of the set of chambers of $\partial_\infty X$ onto itself. We will only be able to define a bijection $\tilde{q}$ between dense subsets of chambers in $\partial_\infty X$.

In order to construct $\tilde{q}$ we use logarithmic flats and logarithmic incidence Weyl chambers. The R–logarithmic flats and the R–logarithmic incidence Weyl chambers with respect to a point x in X (Definitions 3.1.1 and 3.1.3) are analogues in higher rank of the logarithmic geodesics defined by D. Sullivan [Su]. They have the property that at distance $\rho \geq R$ from x they move away of X_0 at most at a distance of order $\log \rho$. D. Sullivan showed [Su, §9, Theorem 6] that, if Γ is a non-uniform lattice in $SO(n, 1)$, almost every geodesic in $\mathbb{H}^n$ is logarithmic. D. Kleinbock and G.A. Margulis [KM] showed that the same thing is true for lattices in higher rank semisimple groups and not only for geodesics but also for maximal flats. They used this result in Diophantine approximation theory. For our purpose, we need a specialized version of this result, in which finite configurations of flats appear and R is fixed. We obtain, with methods similar to those in [KM], a dense set of so-called logarithmic incidence Weyl chambers and logarithmic branching flats (see section 3.1).

In our proof we also use the asymptotic cones of X. An asymptotic cone of a metric space is, in some sense, an "image" of the space seen from infinitely far away. The asymptotic cones of the symmetric space X are Euclidean buildings [KlL, Theorem 5.2.1]. Each asymptotic cone of X contains the limit set, $[X_0]$, of X_0 (see section 2.4 for definitions). The quasi-isometry q induces a bilipschitz map $Q : [X_0] \to [X_0]$. The main property of the logarithmic flats and Weyl chambers is that their limit sets are entirely contained in the limit set $[X_0]$. Therefore, Q may be applied to the limit set of each logarithmic flat, which it sends to a bilipschitz flat in $[X_0]$. Bilipschitz flats in Euclidean buildings locally coincide with fans of Weyl chambers ([KlL], see Proposition 2.3.4 in this paper). By fan of Weyl chambers we mean a finite union of Weyl chambers with common vertex such that their boundaries at infinity form a bilipschitz sphere. A fan whose boundary at infinity coincides with the boundary of a maximal flat is called fan over an apartment.

The previously stated property of bilipschitz flats in Euclidean buildings allows us to go back into the symmetric space X and to conclude that for every logarithmic flat F with respect to x, the image by q of its projection on X_0, $q(\pi_0(F))$, "seen from $q(x)$" approaches a fan of Weyl chambers of vertex $q(x)$ (Proposition 3.3.9). We also obtain that if W is a logarithmic incidence

Weyl chamber of vertex x then $q(\pi_0(W))$ "seen from $q(x)$" approaches a finite union of Weyl chambers of vertex $q(x)$ (Corollary 3.3.10).

Next, we use the ergodicity of the action of an $\mathbb{R}$-torus on $\Gamma\backslash G$ to select a dense subset in the set of logarithmic flats. The flats we choose have the property of branching logarithmically (Definition 3.1.4) with respect to many of their points. We call them *good logarithmic flats.* We show that for such a flat F, $q(\pi_0(F))$ "seen from $q(x)$" approaches a fan over an apartment, and if W is a Weyl chamber in F, $q(\pi_0(W))$ approaches one Weyl chamber (Proposition 4.2.1). This allows us to find an injective map $\tilde{q}$ defined on a dense set of chambers of $\partial_\infty X$.

We also show that if $R > 0$ is fixed then

- all good R-logarithmic flats approach uniformly their associated fans over apartments (Proposition 4.2.6) ;
- if F is a good R-logarithmic flat with respect to x and F' is the maximal flat asymptotic to the fan associated to $q(\pi_0(F))$, then the distance from $q(x)$ to F' is bounded by a constant C depending only on R and on the quasi-isometry constants L and c. (Proposition 4.2.7). As a consequence we obtain that the images by q of big subsets of F are within finite distance from F'. This is a result similar to Theorem 8.1 in [EF], though less general.

Next we extend $\tilde{q}$ to a simplicial isomorphism Φ on $\partial_\infty X$. Our goal is to construct Φ in a way which makes it easy to prove afterwards that Φ is an homeomorphism. Therefore, we do not work directly with the set of good logarithmic flats but we select two dense subsets in it (section 5.1). The first one is the set of *butterfly flats* (Definition 5.1.2). The interest of this type of flat in the setting of the prolongation of $\tilde{q}$ is emphasized by Proposition 5.2.1, which implies that $\tilde{q}$ is uniformly continuous on a full measure set in each chamber star of each wall contained in a butterfly flat. The second set is the set of *pistil flats* (Definition 5.1.3). These flats are butterfly flats which moreover have the property that almost every chamber determines with an opposite chamber in the boundary of the pistil flat a butterfly flat (Corollary 5.1.6).

We construct the simplicial isomorphism Φ on $\partial_\infty X$ starting from the boundary of a pistil flat and the chamber stars of the walls in it. The method is different in the rank two case and in the case of the rank at least 3. The fact that we start from the boundary of a pistil flat instead of the boundary of a butterfly flat allows us to prove that Φ coincides with $\tilde{q}$ on a set of chambers of full measure. The isomorphism Φ is a homeomorphism in the cone topology because it is continuous on the chamber star of each wall in the boundary of the chosen pistil flat (Lemma 5.3.3). By Tits' theorem, Φ defines an isometry $g \in G$. Using Proposition 4.2.7 and the coincidence of Φ with $\tilde{q}$ on a full measure set of chambers, we show that $d(q(\gamma), g(\gamma))$ is bounded by a constant independent of $\gamma \in \Gamma$. The fact that g is in $Comm(\Gamma)$ follows from a result of N. Shah ([Sh], see Theorem 5.4.3 in this paper). With arguments of R. Schwartz [S1] and N. Shah one can derive the other results.

The common features between this proof and Alex Eskin's proof are that they both use Tits' theorem, so in both one can find a construction of a bijection $\tilde{q}$ between dense subsets of spherical chambers of the Tits boundary. Also, in choosing the flats we work with, at a certain point we also use Birkhoff's theorem, and in the first step of the proof of the continuity of Φ, we use a "butterfly argument" similar to an argument used by Alex Eskin in the proof of the Lemma 5.3, [E].

The main differences are that we use asymptotic cone techniques, the class of logarithmic flats and that we prolongate $\tilde{q}$ to the isomorphism Φ in a different way, using results of the theory of buildings. The asymptotic cone techniques allow us to avoid complicated approximation arguments and give good intuitions in several main steps of our argument.

The organization of the paper is as follows:

SECTION 2: We recall some basic facts about buildings and symmetric spaces, about the asymptotic cones and about semisimple groups.

SECTION 3: We consider a non-uniform irreducible lattice Γ in a semisimple group G of rank at least 2. We define R–logarithmic (branching) flats and R–logarithmic incidence Weyl chambers with respect to a point x in X. We show that in almost every point they form dense subsets in the sets of flats through x and of Weyl chambers of vertex x, respectively. We also define the notion of horizon of an infinite set in a CAT(0)-space with respect to a point.

We consider an (L, c)-quasi-isometry $q : \Gamma \to \Gamma$ and its extension to X_0 also denoted by q. We show that for every logarithmic flat F with respect to $x \in X_0$, the set $q(\pi_0(F))$ has the same horizon with respect to $q(x)$ as a fan of Weyl chambers of vertex $q(x)$ and for every logarithmic incidence Weyl chamber W of vertex x, $q(\pi_0(W))$ has the same horizon with respect to $q(x)$ as a finite union of Weyl chambers of vertex $q(x)$.

SECTION 4: We fix a point x in X_0. We choose a dense subset in the set of logarithmic flats with respect to x. The elements in the new dense subset are called good logarithmic branching (good l.b.) flats with respect to x. We show that for a good l.b. flat F, $q(\pi_0(F))$ has the same horizon with respect to $q(x)$ as a fan over an apartment, and if W is a Weyl chamber in F, $q(\pi_0(W))$ has the same horizon as one Weyl chamber (Proposition 4.2.1). We also show the two essential uniformity results Proposition 4.2.6 and Proposition 4.2.7.

Section 5: By Proposition 4.2.1 we have obtained an injective map $\tilde{q}$ defined on a dense set of chambers. We define two new dense sets of maximal flats: the set of butterfly flats and of pistil flats, respectively.

We extend $\tilde{q}$ to a simplicial isomorphism Φ of $\partial_\infty X$ onto itself starting from the boundary of a pistil flat and the chamber stars of the walls in it. Then we show that Φ coincides with $\tilde{q}$ on a dense subset of chambers and that Φ is a homeomorphism in the cone topology. Then Φ defines an isometry g by Tits' theorem. We show that q is at a finite distance from g, and we end the proof of the two theorems.

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2 Preliminaries

2.1 Euclidean cone over a metric space. Let (Y,d) be a metric space of diameter at most π. We call *Euclidean cone over* Y the space $Y \times [0,+\infty)/Y \times \{0\}$ with the metric

$$d\big((x,t),(y,s)\big) = \big(t^2 + s^2 - 2st\cos d_Y(x,y)\big)^{1/2}.$$

We call *δ-Euclidean cone over* Y the space $Y \times [0,\delta)/Y \times \{0\}$ with the same metric.

2.2 Generalities on spherical buildings. We use the notions of spherical building and Euclidean building with their geometric definitions given in [KlL, §3.2, §4.1]. Let Σ be a spherical building of rank r. All its chambers are isometric. Let Δ_{mod} be the spherical simplex in S^{r-1} representing this class of isometry. We call it *the model chamber.* We call codimension 1 walls in Σ *panels.* Let $\mathfrak{M}$ be a wall in Σ. The *chamber star of* $\mathfrak{M}$, $St\,\mathfrak{M}$, is the set of chambers containing $\mathfrak{M}$.

As any spherical building, Σ admits a labelling [Br, IV.1, Proposition 1].

Throughout the whole paper, if Σ has rank r, we will automatically use the labels $\{1, 2, \ldots, r\}$ for the vertices of the chambers and for the opposite panels. Using this labelling one can define a projection $p : \Sigma \to \Delta_{mod}$ on the labelled model chamber. Every isometry of Δ_{mod} onto itself defines a change of labelling of its vertices and panels. By means of the projection p, this new labelling on Δ_{mod} induces a new labelling on Σ. We call this change of labelling an *isometric change of labelling*.

All the apartments of Σ are isomorphic to a finite Coxeter complex S determined by a finite Coxeter group Cox. We call them *the Coxeter complex* and *the Coxeter group associated to* Σ. We call *singular hyperplanes* the hyperplanes of reflection in the apartments. Each singular hyperplane splits any apartment containing it into two parts, called *half-apartments*. By *convex sets in* Σ we designate apartments and intersections of half-apartments. By *convex hull* of a subcomplex contained in an apartment we mean the minimal convex set containing it. We call *singular planes* nonempty intersections of singular hyperplanes. Two chambers are called *opposite* if there is no half-apartment containing both of them. By [T, Proposition 3.25], if two chambers are opposite, there is a unique apartment containing them.

As shown in [KlL, §3.2], any spherical building can be endowed with a metric which makes it a CAT(1)-space and its apartments isometric copies of the Euclidean sphere S^{r-1}. In any apartment, $\mathcal{A}$, it can be then defined a natural opposition isomorphism $op_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Two opposite chambers in the sense of the previous definition are images of one another by the opposition involution in the unique apartment containing them.

Lemma 2.2.1. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two apartments of Σ and $\phi : \mathcal{A} \to \mathcal{A}'$ a simplicial isomorphism. Up to an isometric change of labelling on the target building Σ, ϕ is an isomorphism of labelled simplicial complexes.*

Proof. It is an immediate consequence of [T, §2.6]. □

Let $op_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ be the opposition isomorphism on an apartment. By the previous lemma there exists a permutation, $op : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$, and an isometric change of labelling on Σ induced by it on the target building making $op_{\mathcal{A}}$ an isomorphism of labelled simplicial complexes. The map op does not depend on the chosen apartment $\mathcal{A}$ (easy consequence of the definition of a spherical building).

Two chambers $\mathcal{W}$ and $\mathcal{W}'$ having a panel labelled by i in common are called *i-adjacent*. A *gallery* between two chambers $\mathcal{W}$ and $\mathcal{W}'$ is a finite sequence of chambers, $\mathcal{W}_0 = \mathcal{W}, \mathcal{W}_1, \ldots, \mathcal{W}_k, \mathcal{W}_{k+1} = \mathcal{W}'$, such that for every $j \in \{0, 1, 2, \ldots, k\}$ the chambers $\mathcal{W}_j$ and $\mathcal{W}_{j+1}$ have a panel in common. The *length of the gallery* is the number of chambers composing it. A *nonstammering gallery* is a gallery for which $\mathcal{W}_j \neq \mathcal{W}_{j+1}$, $\forall j \in \{0, 1, 2, \dots, k\}$. If for every j the chambers $\mathcal{W}_j$ and $\mathcal{W}_{j+1}$ are i_j-adjacent, where $i_j \in \{1, 2, \dots, r\}$, we call $(i_0, i_1, i_2, \dots, i_k)$ *the type of the gallery.* A *minimal gallery* between $\mathcal{W}$ and $\mathcal{W}'$ is a gallery of minimal length. The *combinatorial distance* between $\mathcal{W}$ and $\mathcal{W}'$ is the common length of all minimal galleries between them. Convex sets in Σ have the property of containing, together with a pair of chambers, all minimal galleries between them [T, Theorem 2.19].

Let $\mathcal{A}$ be an apartment of Σ and $\mathcal{W}$ a chamber in it. We define the map $\mathrm{retr}_{\mathcal{A},\mathcal{W}} : \Sigma \to \mathcal{A}$ sending each chamber joined to $\mathcal{W}$ by a minimal gallery of type $(i_0, i_1, i_2, \dots, i_k)$ onto the unique chamber in $\mathcal{A}$ joined to $\mathcal{W}$ by a gallery of the same type. The image does not depend on the chosen minimal gallery, and the map $\mathrm{retr}_{\mathcal{A},\mathcal{W}}$ is contractible with respect to the combinatorial distance [T, §3.3–3.6].

LEMMA 2.2.2. *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two chambers in an apartment $\mathcal{A}$ and $\mathcal{D}$ their convex hull. Let $\varpi = op_{\mathcal{A}} \circ \mathrm{retr}_{\mathcal{A},\mathcal{W}_1}$. We denote $Op_{\mathcal{W}_1}(\mathcal{W}_2) = \{\mathcal{W} \in \Sigma \mid \varpi(\mathcal{W}) = \mathcal{W}_2\}$.*

(i) *The set $Op_{\mathcal{W}_1}(\mathcal{W}_2)$ does not depend on the choice of the apartment $\mathcal{A}$ containing $\mathcal{W}_1$ and $\mathcal{W}_2$. The chambers $\mathcal{W}$ in $Op_{\mathcal{W}_1}(\mathcal{W}_2)$ are the chambers opposite to $\mathcal{W}_2$ such that $\mathcal{W}_1$ is contained in a minimal gallery from $\mathcal{W}$ to $\mathcal{W}_2$.*

(ii) *The map associating to each $\mathcal{W} \in Op_{\mathcal{W}_1}(\mathcal{W}_2)$ the apartment determined by it and $\mathcal{W}_2$ is a bijection onto the set of apartments containing $\mathcal{D}$.*

Proof. Both (i) and (ii) are consequences of the definition of a building, of the properties of opposite chambers and of [T, Lemma 4.2]. □

A *bilipschitz sphere* in Σ is a finite union of chambers which is the image of a bilipschitz embedding of S^{r-1}.

LEMMA 2.2.3. *Let Σ be a spherical building. Suppose all its apartments have $p_0 + 1 = 2q_0$ chambers.*

(i) *Any bilipschitz sphere in Σ has at least $p_0 + 1$ chambers.*

(ii) *A bilipschitz sphere in Σ which has exactly $p_0 + 1$ chambers is an apartment.*

(iii) *If Σ has rank two, then any two chambers between which there exists a nonstammering gallery of length $q_0 + 1$ are opposite.*

Proof. (i) Let BS be a bilipschitz sphere, $\mathcal{W}$ a chamber in BS and $\mathcal{A}$ an apartment containing $\mathcal{W}$. We consider the retraction retr=$\mathrm{retr}_{\mathcal{A},\mathcal{W}}$. For every $\mathcal{W}'$ chamber in $\mathcal{A}$ there is a minimal gallery $\mathcal{G}$ of type $(i_0, i_1, \dots, i_k)$ joining $\mathcal{W}'$ to $\mathcal{W}$. In BS we can obviously construct a gallery $\mathcal{G}'$ of the same type starting from $\mathcal{W}$, and $\mathcal{G}'$ will also be minimal [R, chapter 3, (3.1)(iv)]. Then $\mathrm{retr}(\mathcal{G}') = \mathcal{G}$. Thus $\mathrm{retr}(BS) = \mathcal{A}$.

(ii) In this case $\mathrm{retr}|_{BS}$ is also injective on the set of chambers, so an isomorphism of simplicial complexes.

(iii) is an easy consequence of the fact that in a building of rank two there do not exist cycles of length smaller than $p_0 + 1$. $\square$

2.3 Symmetric spaces and Euclidean buildings.

2.3.A The Tits boundary. Let (X, d) be a CAT(0)-space. We consider its boundary at infinity, $\partial_\infty X$ [B]. For every x in X and α in $\partial_\infty X$, there exists a unique geodesic ray r_α satisfying $r_\alpha(0) = x$, $r_\alpha(\infty) = \alpha$. We shall sometimes denote it by $[x, \alpha)$. If X is a symmetric space, the map $\varphi_x : \partial_\infty X \to T_x X$, $\varphi_x(\alpha) = \dot{r}_\alpha(0)$ is a bijection. The topology on $\partial_\infty X$ that makes φ_x a homeomorphism is called *the cone topology.* The cone topology does not depend on the chosen point x.

Let $[x, a)$ and $[x, b)$, $a, b \in X$, be nondegenerate geodesic segments. We consider the triangle $\bar{x}\bar{a}\bar{b}$ in $\mathbb{R}^2$ having the same edge lengths as the geodesic triangle with vertices x a and b in X. We denote the angle obtained in $\bar{x}\bar{a}\bar{b}$ in the vertex $\bar{x}$ by $\tilde{\angle}_x(a, b)$ and we call it *the comparison angle between a and b seen from x.*

Suppose now that $a, b \in X \bigsqcup \partial_\infty X$. We denote by a_t the element on the geodesic segment or ray $[x, a)$ at distance t of x and by b_s the element on $[x, b)$ at distance s of x. *The angle between the geodesic segments or rays* $[x, a)$ *and* $[x, b)$ is defined as

$$\angle_x(a, b) := \lim_{t,s \to 0} \tilde{\angle}_x(a_t, b_s)\,.$$

If X is a Riemannian manifold, $\angle_x(a, b)$ coincides with the angle between the tangent vectors of the two geodesic segments or rays in x. The comparison angle $\tilde{\angle}_x(a, b)$ is in general strictly larger than the angle $\angle_x(a, b)$ [KlL, Triangle Filling Lemma 2.1.4]. Since an angle in an Euclidean triangle depends on the ratios between the lengths of the edges, the comparison angle gives information on the order of $d(a, b)$ compared to the order of $d(x, a)$ and $d(x, b)$ rather than on the angle between the segments $[x, a)$ and $[x, b)$.

The *Tits metric* on $\partial_\infty X$ is the metric defined by

$$d_T(\alpha, \beta) := \sup_{x \in X} \angle_x(\alpha, \beta)\,, \quad \forall \alpha, \beta \in \partial_\infty X\,.$$

Let X be a symmetric space or an Euclidean building of rank $r \geq 2$. The boundary $\partial_\infty X$ with the Tits metric is a spherical building of rank r ([Mos, Chapters 15,16], [BGS, Appendix 5], [KlL, §4.2.1]). It follows that $\partial_\infty X$ admits a labelling by means of which one can define a projection $p : \partial_\infty X \to \Delta_{mod}$ on a model chamber. If X is a symmetric space, for every x in X one can also define a projection $p_x : T_x X \to \Delta_{mod}$ by $p_x = p \circ \varphi_x^{-1}$.

Let F be a maximal flat (an apartment) in X. We call a *singular hyperplane in* F a hyperplane in F which appears as the intersection of F with another maximal flat (apartment). We call a *singular plane in* F a nonempty intersection of singular hyperplanes. A singular hyperplane splits each apartment containing it into two *half-apartments*. We call *Weyl polytope* an intersection of half-apartments. We notice that with this definition we include in the class of Weyl polytopes singular (hyper)planes and polytopes in them. The nonempty intersection of two apartments is always a Weyl polytope. We say that a singular plane of dimension d *supports* a Weyl polytope if they are both contained in the same apartment F and the plane intersects the boundary of the polytope in a set of dimension d.

Let $\mathcal{M}$ be a wall of a Weyl chamber of vertex x. The *chamber star of* $\mathcal{M}$, *St* $\mathcal{M}$, is the set of Weyl chambers of vertex x containing $\mathcal{M}$.

Let F be a maximal flat (an apartment) in X, H a singular (hyper)plane in F and W a Weyl chamber in X. We denote by $F(\infty), H(\infty)$ and $W(\infty)$ the boundaries at infinity of F, H and W respectively, considered as subsets of $\partial_\infty X$. The boundary at infinity of a maximal flat (an apartment) determines it: $F_1(\infty) = F_2(\infty)$ implies $F_1 = F_2$. The boundaries of maximal flats (apartments) in X are precisely the apartments of $\partial_\infty X$, the boundaries of Weyl chambers are the chambers of $\partial_\infty X$. If H is a singular (hyper)plane in a flat F, $H(\infty)$ is a singular (hyper)plane in $F(\infty)$.

A sequence $W_0, W_1, W_2, \dots, W_{k+1}$ of Weyl chambers of vertex x is called a *minimal gallery of type* $(i_0, i_1, i_2, \dots, i_k)$ *from* W_0 *to* W_{k+1} if $W_0(\infty), W_1(\infty), W_2(\infty), \dots, W_{k+1}(\infty)$ is a minimal gallery of type $(i_0, i_1, i_2, \dots, i_k)$ in $\partial_\infty X$.

Definition 2.3.1. *We call fan of vertex x a union of Weyl chambers of vertex x, $\cup_{i=0}^{p} W_i$, such that $\cup_{i=0}^{p} W_i(\infty)$ is a bilipschitz sphere in $(\partial_\infty X, d_T)$.*

If $\cup_{i=0}^{p} W_i(\infty)$ is moreover an apartment, $\cup_{i=0}^{p} W_i$ is called a fan over an apartment.

If X is a symmetric space, every isometry g of X induces a map on the boundary, $\partial g : \partial_\infty X \to \partial_\infty X$, which is a simplicial isomorphism with respect to the spherical building structure and an homeomorphism with respect to the cone topology. The following converse statement plays an essential part in our reasoning.

Theorem 2.3.2 (Tits, [T]). *Let X be a nonpositively curved symmetric space with no rank one factors. Let $\Phi : \partial_\infty X \to \partial_\infty X$ be a simplicial isomorphism with respect to the spherical building structure and an homeomorphism with respect to the cone topology. Then there exists a unique isometry g of X such that $\Phi = \partial g$.*

2.3.B The building of directions. Let $\mathbf{K}$ be a CAT(0)-space and x a point in $\mathbf{K}$. Two nondegenerate geodesic segments or rays starting from $x, [x,a)$ and $[x,b)$, *have the same direction in* x if $\angle_x(a,b) = 0$. The *space of directions in* x *of* $\mathbf{K}$ is the space of equivalence classes of nondegenerate geodesic segments and rays starting from x with respect to the equivalence relation "same direction in x". We denote it by $\Sigma_x\mathbf{K}$. We denote the equivalence class containing the segment or the ray $[x,a)$ by $\overline{xa}$. The space $\Sigma_x\mathbf{K}$ is a metric space with the metric $d_x(\overline{xa}, \overline{xb}) = \angle_x(a,b)$. If $\mathbf{K}$ is a symmetric space, we identify $(\Sigma_x\mathbf{K}, d_x)$ with the unit tangent vector space in x, by identifying each equivalence class $\overline{xa}$ with the common unit tangent vector.

Proposition 2.3.3 [KlL]. *If $\mathbf{K}$ is a homogeneous Euclidean building then, for every x in $\mathbf{K}$, $(\Sigma_x\mathbf{K}, d_x)$ is a spherical building. Its apartments are the sets of directions $F_x = \{\overline{xa} : [x,a] \subset F\}$, where F is an apartment through x, and its chambers are $W_x = \{\overline{xa} : [x,a] \subset W\}$, where W is a Weyl chamber of vertex x.*

Let $\mathbf{K}$ be a homogeneous Euclidean building. From the previous proposition it follows that $(\Sigma_x\mathbf{K}, d_x)$, for every $x \in \mathbf{K}$, and $(\partial_\infty\mathbf{K}, d_T)$ have the same associated Coxeter complex S and Coxeter group Cox. We call them *the Coxeter complex and the Coxeter group associated to* $\mathbf{K}$. As in the symmetric space case, for every $x \in \mathbf{K}$ we can define a map $\varphi_x : \partial_\infty\mathbf{K} \to \Sigma_x\mathbf{K}$ by $\varphi_x(\alpha) = \overline{x\alpha}$. The definition of the Tits metric implies that φ_x is a contraction. In general, it is not a bijection. If we fix a labelling on $\partial_\infty\mathbf{K}$, for every $x \in \mathbf{K}$ we may choose a compatible labelling on $\Sigma_x\mathbf{K}$. That is, if $p_x : \Sigma_x\mathbf{K} \to \Delta_{mod}$ is the corresponding projection and $P : \partial_\infty\mathbf{K} \to \Delta_{mod}$ is the projection corresponding to the labelling on $\partial_\infty\mathbf{K}$, then $p_x \circ \varphi_x = P$. In the sequel we shall always suppose that each time we have chosen a labelling on $\partial_\infty\mathbf{K}$, we have correspondingly chosen compatible labellings on all $\Sigma_x\mathbf{K}$, $x \in \mathbf{K}$. Then for every geodesic ray $[x_0,\alpha)$ in $\mathbf{K}$, for all $x \in [x_0,\alpha)$ we have $p_x(\overline{x\alpha}) = P(\alpha) = \theta$. We call θ *the direction of the ray* $[x_0,\alpha)$.

Since every geodesic segment $[x,y]$ can be prolongated to a ray, the previous reasoning also implies that for every $t \in (x,y)$ we have that $p_t(\overline{ty}) = p_x(\overline{xy}) = \theta$. Then θ is called *the direction of the segment* $[x,y]$. In both cases, if θ is in the interior of Δ then the ray or the segment are called *regular*, otherwise they are called *singular*.

2.3.C Bilipschitz flats in Euclidean buildings. Let $\mathbf{K}$ be an Euclidean building of rank r. We call *L-bilipschitz flat* in $\mathbf{K}$ the image of an L-bilipschitz embedding of $\mathbb{R}^r$ in $\mathbf{K}$. B. Kleiner and B. Leeb proved that a bilipschitz flat in an Euclidean building has the following properties:

PROPOSITION 2.3.4 [KlL]. *Let BF be an L-bilipschitz flat in $\mathbf{K}$. Then*

(i) *There exists a finite set of apartments, $F_1, F_2, \dots, F_m$ in $\mathbf{K}$, such that $BF \subset \cup_{i=1}^m F_i$. Moreover the number of apartments m is bounded by a constant M depending only on L.*

(ii) *For every point y in BF there exists $\delta > 0$ and $W_0, W_1, \dots, W_\ell$ Weyl chambers of vertex y such that*

$$BF \cap B(y,\delta) = \bigcup_{i=0}^{\ell} W_i \cap B(y,\delta)\,.$$

Moreover, $\cup_{i=0}^{\ell} W_i \cap B(y,\delta)$ is a δ-Euclidean cone over a bilipschitz sphere in $\Sigma_y \mathbf{K}$ and the number of Weyl chambers ℓ is bounded by a constant M' depending only on L.

(iii) *Let x be a point in $\mathbf{K}$ and y be a point in BF. The geodesic segment $[x,y]$ can be prolongated to a geodesic ray $[x,\alpha) = [x,y] \cup [y,\alpha)$ such that $[y,\alpha) \subset BF$.*

See [KlL, Corollary 6.2.3, Lemma 7.2.3, Corollary 7.2.4] for proofs of these properties.

From the previous properties of bilipschitz flats we may deduce that for every bilipschitz flat BF and for every point x the set of points in BF which are in regular directions starting from x is dense in BF.

COROLLARY 2.3.5. *Let BF be an L-bilipschitz flat and x a point in $\mathbf{K}$. We define the map $P_x : \mathbf{K} \setminus \{x\} \to \Delta_{mod}$ by $P_x(y) = p_x(\overline{xy})$. The set $BF \cap P_x^{-1}(Int\, \Delta_{mod})$ is dense in BF.*

Proof. We reason by contradiction and suppose that for a certain point z in BF there exists $\varepsilon > 0$ such that $P_x(B(z,\varepsilon) \cap BF) \subset \partial \Delta_{mod}$. We consider the restriction $P'_x : B(z,\varepsilon) \cap BF \to \partial \Delta_{mod}$. Let θ be a fixed point in the image of P'_x. For every y in $(P'_x)^{-1}(\theta)$ the geodesic segment $[x,y]$ can be prolongated to a ray $[x,\alpha)$ as in Proposition 2.3.4 (iii). Then

$P(\alpha) = \theta$ and $\alpha \in \cup_{i=1}^{m} F_i(\infty)$, where $F_1, F_2, \dots, F_m$ is the finite set of apartments covering BF. Then there is a finite set of possibilities for α, $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$. We have thus proved that $(P'_x)^{-1}(\theta) \subset \cup_{j=1}^{k} [x, \alpha_j)$. This implies that the topological dimension of $(P'_x)^{-1}(\theta)$ is at most 1. By [N, Theorem III.6], the topological dimension of $B(z, \varepsilon) \cap BF$ is at most dim $\partial\, \Delta_{mod} + 1$, so at most $r - 1$. But $B(z, \varepsilon) \cap BF$ contains the bilipschitz image of a small ball in $\mathbb{R}^r$, so its dimension is at least r, which gives a contradiction. $\square$

2.4 Ultralimits and asymptotic cones.

2.4.A In order to define ultralimits and asymptotic cones we need nonprincipal ultrafilters. A *nonprincipal ultrafilter* is a probability measure $\omega : P(\mathbb{N}) \to \{0, 1\}$, finitely additive, such that $\omega(A) = 0$ for all $A \subset \mathbb{N}$, A finite. Nonprincipal ultrafilters always exist [Dr, §2.1]. A nonprincipal ultrafilter ω "chooses" one among a finite set of possibilities, and this will be the only possibility that will "count" in the reasonings made ω-almost surely. Indeed, for every partition $\mathbb{N} = \bigsqcup_{i=1}^{k} A_i$ there exists i_0, $1 \le i_0 \le k$, such that $\omega(A_{i_0}) = 1$ and $\omega(A_i) = 0$, $\forall i \neq i_0$.

In a topological space X the *ω-limit* of a sequence (a_n) is an element a such that for every neighborhood V of a, $\omega(\{n \in \mathbb{N} \mid a_n \in V\}) = 1$. The convergence of all ultrafilters characterizes compact sets [Bo, I.9.1]. Hence, every sequence in a compact set has an ω-limit. In particular, every bounded numerical sequence has an ω-limit.

Let (X_n, d_n) be a sequence of pointed metric spaces with x_n fixed basepoints in X_n. We consider the set of sequences

$$\mathfrak{S} = \left\{ (y_n) \;\middle|\; y_n \in X_n, \forall n \in \mathbb{N}, \text{ and } (d_n(x_n, y_n))_{n \in \mathbb{N}} \text{ is a bounded sequence} \right\},$$

on which we define the equivalence relation

$$(y_n) \sim (z_n) \Leftrightarrow \lim_{\omega} d_n(y_n, z_n) = 0\,.$$

The quotient space $\mathfrak{S}/\sim$ is called *the based ultralimit of the sequence of pointed metric spaces* (X_n, d_n, x_n) and it is denoted either by X_ω or by $\lim_\omega (X_n, d_n, x_n)$. We shall denote the equivalence class containing the sequence (y_n) either by $[y_n]$ or by y_ω. The space X_ω is a metric space with the metric

$$d(y_\omega, z_\omega) = \lim\nolimits_\omega d_n(y_n, z_n),$$

and it also has a basepoint $x_\omega = [x_n]$. As a metric space X_ω is complete [KlL, Lemma 2.4.2].

If the sequence of pointed metric spaces (X_n, d_n, x_n) converges to a pointed metric space (Y, d, y) in the modified Hausdorff metric [G1, §6], then for every nonprincipal ultrafilter ω, the based ultralimit (X_ω, d, x_ω) is isometric to (Y, d, y) [KlL, Lemma 2.4.3]. In particular the same thing happens if X_n are all compact and the sequence converges in the Hausdorff metric to a compact metric space Y. But in general, sequences of metric spaces do not have limits in the (modified) Hausdorff metric.

A subset $A \subset X_\omega$ is called *limit set* if there exists a sequence of sets $A_n \subset X_n$ such that $A = \{[x_n]\ ;\ x_n \in A_n\ \omega\text{-a.s.}\}$. We set $A = [A_n]$.

REMARK 2.4.1. *A limit set is a closed subset of the based ultralimit.*

Proof. Let A be a limit set, $A = [A_n]$. Let $a_\omega = [a_n] \in A$. The metric space (A, d, a_ω) is the ω-based ultralimit of the sequence (A_n, d_n, a_n), hence it is complete. This implies that it is closed in (X_ω, d). □

We now define the asymptotic cone of a metric space (X, d). It is meant to be a kind of limit space representing X "seen from infinitely far away". Let ω be a nonprincipal ultrafilter, $f_0 : \mathbb{N} \to X$ a sequence called *sequence of observation centers*, and $(\iota_n) \in \mathbb{R}^{\mathbb{N}}$ a numerical sequence, $\iota_n \to \infty$, called *sequence of scalars. The ω-asymptotic cone of X with respect to the observation centers $(f_0(n))$ and scalars (ι_n)*, denoted by $Con_\omega(X, f_0, (\iota_n))$ is the ω-based ultralimit of the sequence of metric spaces $\left(X, \frac{1}{\iota_n} d, f_0(n)\right)$. For generic metric spaces X, the asymptotic cone depends on the chosen nonprincipal ultrafilter, as well as on the observation points and on the scalars. So generic metric spaces have an infinity of asymptotic cones associated to them.

The notion of asymptotic cone was introduced by M. Gromov ([G1,2]), L. van den Dries and A.J. Wilkie ([DW]). Other references are [KlL] and [Dr].

Let $\mathbb{R}^k$ be an Euclidean space with the Lebesgue measure ν. Let $\mathcal{B}$ be the σ-algebra of Borel sets and $\mathcal{B}^0$ its completion, the σ-algebra of measurable sets. Any asymptotic cone, $Con_\omega(\mathbb{R}^k, (x_n), (\iota_n))$, is isometric to $\mathbb{R}^k$. We denote such a cone by $\mathbb{R}^k_\omega$. Since $\mathbb{R}^k$ has plenty of homotheties, we can also see $\mathbb{R}^k_\omega$ as the based ultralimit $\lim_\omega(\mathbb{R}^k, x_n)$. For every sequence of sets $A_n \subset \mathbb{R}^k$ such that $A_n \in \mathcal{B}^0$ and $d(x_n, A_n) \le c_A \iota_n$, we can consider the limit set $A_\omega = [A_n]$ and we can define a measure of A_ω by

$$\nu_\omega(A_\omega) = \lim_\omega \frac{\nu(A_n)}{(\iota_n)^k}\,.$$

Equivalently, by looking at $\mathbb{R}^k_\omega$ as $\lim_\omega(\mathbb{R}^k, x_n)$, we can consider sequences $A_n \subset \mathbb{R}^k$ such that $A_n \in \mathcal{B}^0$ and $d(x_n, A_n) \le c_A$, limit sets

$A_\omega = [A_n]$, and we can define $\nu_\omega(A_\omega) = \lim_\omega \nu(A_n)$. Let $\mathcal{B}_\omega := \{A_\omega = [A_n] \mid A_n \in \mathcal{B}\ \omega\text{-almost surely}\}$ and $\mathcal{B}^0_\omega := \{A_\omega = [A_n] \mid A_n \in \mathcal{B}^0\ \omega\text{-almost surely}\}$. Both are algebras of sets. We show that $\mathcal{B}_\omega \subset \mathcal{B}$, that $\mathcal{B}^0_\omega \subset \mathcal{B}^0$ and that ν_ω is a restriction of ν. Let $x_\omega = [x_n]$. Obviously $A_\omega = [A_n] = [\overline{A}_n]$, where $\overline{A}_n$ is the topological closure of A_n, so we may consider only limits of closed sets. Since $A_\omega = \cup_{m\in\mathbb{N}} A_\omega \cap B(x_\omega, m)$, we may restrict ourselves to bounded limit sets, limits of bounded sets. So in the end we may consider only limits of compact sets. By [G1, §6], any sequence A_n of uniformly bounded compact subsets of $\mathbb{R}^k$ has as ω-limit in the Hausdorff metric a compact subset $A \subset \mathbb{R}^k$. By [KlL, Lemma 2.4.3], A_ω is isometric to A. We conclude that $\mathcal{B}_\omega \subset \mathcal{B}$ and that $\nu_\omega(A_\omega) = \nu(A) = \nu(A_\omega)$. The inclusion $\mathcal{B}_\omega \subset \mathcal{B}$ is strict since open balls are not in $\mathcal{B}_\omega$. The previous reasoning also implies that in the case of unbounded closed sets $A_\omega = [A_n]$, A_ω is isometric to the ω-limit of (A_n) with respect to the modified Hausdorff metric and that $\nu_\omega(A_\omega) = \nu(A_\omega)$ (see also [G1, §6, Compactness Criterion]). Thus, the measure ν_ω is well defined. Moreover $\nu_\omega = \nu|_{\mathcal{B}_\omega}$. We can easily extend our conclusions to $\mathcal{B}^0_\omega$ and $\mathcal{B}^0$.

2.4.B We have the following result

Theorem 2.4.2 ([KlL]). *Let X be a symmetric space or an Euclidean building. Every asymptotic cone of X, $\mathbf{K} = Con_\omega(X, (x_n), (\iota_n))$, is a homogeneous Euclidean building having as apartments limit sets of sequences of maximal flats in X. Moreover, the singular (hyper)planes, the Weyl chambers and their walls and the Weyl polytopes are limit sets of sequences of objects of the same kind in X. In particular, $\partial_\infty \mathbf{K}$ and $\partial_\infty X$ are two spherical buildings with the same model chamber Δ_{mod}, the same Coxeter complex and the same Coxeter group.*

If we fix a labelling on the spherical building $\partial_\infty X$, it induces a labelling on $\partial_\infty \mathbf{K}$. We can then define a projection $P : \partial_\infty \mathbf{K} \to \Delta_{mod}$ with respect to this labelling. In the sequel we shall always consider the boundaries at infinity of the asymptotic cones with the labellings induced by the ones on $\partial_\infty X$.

In this particular case of Euclidean buildings (obtained as asymptotic cones), we shall sometimes call the apartments *maximal flats*.

In the sequel we shall also need the following simple remark.

Lemma 2.4.3. *Let X be a symmetric space or an Euclidean building and F^0 and F^1 be two maximal flats in it whose boundaries at infinity intersect in the (convex) set $\mathcal{C}$. Then in any asymptotic cone of X with fixed observation point x, $Con_\omega(X, x, \iota_n)$, the limit maximal flats F^0_ω and*

F^1_ω intersect in an Euclidean cone of vertex $x_\omega = [x]$ over $\mathcal{C}$.

Proof. Let $Con_\omega(X, x, \iota_n)$ be an asymptotic cone of X with fixed observation point x and x_i be the projection of x on F^i, $i = 0, 1$. The Euclidean cone in F^0 over $\mathcal{C}$ of vertex x_0 and the Euclidean cone in F^1 over $\mathcal{C}$ of vertex x_1 are at Hausdorff distance $d(x_1, x_2)$. So in the asymptotic cone their limit sets coincide with an Euclidean cone over $\mathcal{C}$ contained both in F^0_ω and F^1_ω, of vertex $[x_0] = [x_1] = [x]$.

Now let y_ω be an arbitrary point in $F^0_\omega \cap F^1_\omega$ at distance $\delta > 0$ from x_ω. Then $y_\omega = [y^0_n] = [y^1_n]$, where (y^i_n) is a sequence of points in F^i, $i = 0, 1$, with $\lim_\omega d(x, y^i_n)/\iota_n = \delta$. The sequence (y^i_n) will then have as ω-limit in the compactification of X a point α_i in the boundary of F^i, $i = 0, 1$.

The equality $[y^0_n] = [y^1_n]$ implies that $\lim_\omega d(y^0_n, y^1_n)/\iota_n = 0$, so also that $\lim_\omega d(y^0_n, y^1_n)/d(x, y^i_n) = 0$, $i = 0, 1$. Then the ω-limit of the angle $\angle_x(y^0_n, y^1_n)$ is zero. This implies that $\alpha_0 = \alpha_1 = \alpha$ which is a point in $\mathcal{C} = F^0(\infty) \cap F^1(\infty)$. Now if we look in F^0 at the geodesic ray $[x_0, \alpha)$ and at the geodesic segments $[x_0, y^0_n]$, the ω-limit of the angle between them must be 0. Then, if z_n is the projection of each y^0_n on $[x_0, \alpha)$, we have that $\lim_\omega d(z_n, y^0_n)/d(x, y^0_n) = 0$, which implies $\lim_\omega d(z_n, y^0_n)/\iota_n = 0$. Thus, in the asymptotic cone, y_ω coincides with $[z_n]$, which is a point in the limit set of $[x_0, \alpha)$. It follows that y_ω is contained in the Euclidean cone of vertex x_ω over $\mathcal{C}$ which the maximal flats F^0_ω and F^1_ω had in common. □

2.5 Preliminaries on semisimple groups. Let G be a semisimple group of rank $r \geq 2$, with finite center and without compact factors and let K be a maximal compact subgroup. We consider G and K as fixed and verifying these conditions for the rest of the paper. Let $X = G/K$ be the symmetric space associated to G and $x_0 = eK \in X$. For any point $x \in X \setminus \{x_0\}$, we denote by K_x the maximal compact subgroup fixing it.

In the sequel every group H is considered with its Haar measure μ_H and with a left invariant metric d_H on it (the canonical one, each time it is the case). Every homogeneous space H/H_1 is considered with its H-invariant natural measure and metric. Whenever a set of maximal flats or Weyl chambers is parametrized in a one-one fashion by a group or a homogeneous space, we automatically consider it endowed with the induced measure and metric.

We introduce some notation and we choose a finite family of small unipotents in G which will be useful in proving the density results in section 3.2. The reference for the general theory of semisimple groups and symmetric spaces is [H].

2.5.A The group G acts on itself by conjugacy, so we may define the group homomorphism

$$a : G \to Aut(G)\,, \quad a(g_0)(g) = g_0 g g_0^{-1}\,.$$

Let $\mathfrak{g}$ be the Lie algebra of G which we may identify, as a vector space, with the tangent space T_eG. The homomorphism a induces the adjoint representation $Ad : G \to GL(\mathfrak{g}),\ Ad(g) = d_e a(g)$.

Let s_0 be the geodesic symmetry in X with respect to the basepoint x_0. Then $Ad(s_0)$ is an involution of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the decomposition of $\mathfrak{g}$ as sum of proper spaces of $Ad(s_0)$, $\mathfrak{p}$ corresponding to the proper value -1 and $\mathfrak{k}$ to 1. It is well known that $\mathfrak{k}$ is the Lie algebra of K and that $\mathfrak{p}$ can be identified to $T_{x_0}X$ as a vector space. We choose (and fix) a maximal $\mathbb{R}$-torus A in G, whose Lie algebra $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then Ax_0 is a maximal flat in X which we denote by F_0. We also denote $F_0(\infty)$ by $\mathcal{F}_0$. We denote by $Z(A)$ and $N(A)$ the centralizer and the normalizer of A in G, respectively.

We have the following decomposition of the Lie algebra as a direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha\in\Xi}\mathfrak{g}_\alpha$, where Ξ is the set of roots with respect to A, $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid \forall a \in A,\ Ad(a)(Y) = \alpha(a)Y\}$, and $\mathfrak{g}_0 = \{Y \in \mathfrak{g} \mid \forall a \in A,\ Ad(a)(Y) = 0\}$.

Notation. For every $\alpha \in \Xi$ we denote $A_\alpha = \operatorname{Ker}\alpha$, $\mathfrak{D}_\alpha^+ = \{a \in A \mid \alpha(a) \geq 1\}$ and $K_\alpha = K \cap Z(A_\alpha)$. For every $\Lambda \subset \Xi$ we denote $A_\Lambda = \cap_{\alpha\in\Lambda} A_\alpha$, $\mathfrak{D}_\Lambda^+ = \cap_{\alpha\in\Lambda}\mathfrak{D}_\alpha^+$ and $K_\Lambda = K \cap Z(A_\Lambda)$. We also denote $K_A = K \cap Z(A)$.

We recall that $A_\alpha x_0$ is a singular hyperplane in F_0, which we denote by H_0^α, $\mathfrak{D}_\alpha^+ x_0$ a half of F_0 determined by this hyperplane, which we denote by D_α^+ and that K_α/K_A parametrizes the family of maximal flats containing H_0^α. Similarly $A_\Lambda x_0$ is a singular plane of codimension card Λ, which we denote by H_0^Λ and $\mathfrak{D}_\Lambda^+ x_0$ is a convex set which appears as intersection of the half-flats D_α^+, $\alpha \in \Lambda$, and which we denote D_Λ^+. Also K_Λ/K_A parametrizes the family of maximal flats containing H_0^Λ.

The finite group $(K \cap N(A))/K_A$ coincides with the Coxeter group of the Coxeter complex determined on $F_0(\infty)$ by the boundaries at infinity of singular hyperplanes. We choose (and fix) $e = \sigma_0, \sigma_1, \dots, \sigma_{p_0}$, a system of representatives for $(K \cap N(A))/K_A$.

We fix a Weyl chamber $\triangleleft A_0$ in A. Let $\Delta_0 = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be the fundamental set of roots and let Ξ_+ be the set of positive roots corresponding to it. The orbit $\triangleleft A_0 x_0$ is a Weyl chamber in X that we denote by W_0, and we denote $W_0(\infty)$ by $\mathcal{W}_0$. If $\mathfrak{u}_0 = \oplus_{\alpha\in\Xi_+}\mathfrak{g}_\alpha$, then the group $U_0 = \exp\mathfrak{u}_0$ is the unipotent group associated to the Weyl chamber $\triangleleft A_0$. The group U_0 acts simply transitively on the set of flats of X asymptotic to W_0. The flats asymptotic to W_0 correspond to the apartments in $\partial_\infty X$ containing $\mathcal{W}_0$, and this set of apartments can be identified with the set of chambers in $\partial_\infty X$ opposite to $\mathcal{W}_0$. We conclude that U_0 acts simply transitively on the set of chambers opposite to $\mathcal{W}_0$.

We consider all the Lie subalgebras of $\mathfrak{g}$ of the form $\mathfrak{u}_\Psi = \oplus_{\alpha\in\Psi}\mathfrak{g}_\alpha$, where $\Psi \subset \Xi_+$, $\Psi \neq \Xi_+$, and $U_\Psi = \exp \mathfrak{u}_\Psi$. The subspace $\oplus_{\alpha\in\Psi}\mathfrak{g}_\alpha$ is a Lie subalgebra if and only if Ψ is closed with respect to the addition.

NOTATION. Let $\alpha \in \Xi_+$ and $\Psi_\alpha = \Xi_+ \cap \{m\alpha \mid m \in \mathbb{N}\}$. We denote $\mathfrak{u}_{\Psi_\alpha}$ by $\mathfrak{u}_\alpha$ and U_{Ψ_α} by U_α.

The group U_Ψ acts simply transitively on the set of flats asymptotic to D^+_Ψ, so on the set of apartments containing $D^+_\Psi(\infty)$.

In the sequel we choose and fix a chamber $\mathcal{W}_1$ in $\mathcal{F}_0$ such that $D^+_\Psi(\infty)$ is the convex hull of $\mathcal{W}_0$ and $\mathcal{W}_1$. By Lemma 2.2.2, the set of apartments containing $D^+_\Psi(\infty)$ can be identified with the set of chambers $Op_{\mathcal{W}_0}(\mathcal{W}_1)$. So U_Ψ acts simply transitively on $Op_{\mathcal{W}_0}(\mathcal{W}_1)$. In particular the group U_α acts simply transitively on the set of apartments containing the half-apartment D^+_α. Suppose $\alpha \in \Delta_0$. Then this set of apartments can be identified with the set of chambers having in common with $\mathcal{W}_0$ one panel contained in $H_0^\alpha(\infty)$. So on $\partial_\infty X$ the action of an element of U_α can be seen as fixing the half-apartment $D^+_\alpha(\infty)$ and "rotating the other half-apartment of $\mathcal{F}_0$ around $H_0^\alpha(\infty)$".

We consider the map retr=$\mathrm{retr}_{\mathcal{F}_0,\mathcal{W}_0} : \partial_\infty X \to \mathcal{F}_0$ and $op_{\mathcal{F}_0} : \mathcal{F}_0 \to \mathcal{F}_0$. For every $\mathcal{W} \in Op_{\mathcal{W}_0}(\mathcal{W}_1)$, $\mathrm{retr}(\mathcal{W}) = \mathrm{op}_{\mathcal{F}_0}(\mathcal{W}_1) = \mathcal{W}_1^{op}$. Let $\mathcal{W}_0 = \widetilde{\mathcal{W}}_0, \widetilde{\mathcal{W}}_1, ..., \widetilde{\mathcal{W}}_{m+1} = \mathcal{W}_1^{op}$ be a minimal gallery and $W_0 = \widetilde{W}_0, \widetilde{W}_1, ..., \widetilde{W}_{m+1} = W_1^{op}$ the corresponding minimal gallery of Weyl chambers of vertex x_0 in F_0. Let α_j be the root such that $D^+_{\alpha_j}$ contains $\widetilde{W}_j$ and does not contain $\widetilde{W}_{j+1}$, $j \in \{0, 1, 2, \dots m\}$. Then, by [R, §4], the map

$$\begin{array}{ccc} U_{\alpha_0} \times U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_m} & \longrightarrow & U_\Psi \\ (u_{\alpha_0}, u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_m}) & \longrightarrow & u_{\alpha_0} \cdot u_{\alpha_1} \cdot u_{\alpha_2} \cdots u_{\alpha_m} \end{array}$$

is a diffeomorphism. The significance of this diffeomorphism is that the chambers in $Op_{\mathcal{W}_0}(\mathcal{W}_1)$ can be parametrized in two different ways. The first one is by means of the group U_Ψ. The second one can be seen as follows. Every chamber $\mathcal{W} \in Op_{\mathcal{W}_0}(\mathcal{W}_1)$ may be joined to $\mathcal{W}_0$ by a minimal gallery $\mathcal{W}_0 = \widetilde{\mathcal{W}}'_0, \widetilde{\mathcal{W}}'_1, \dots, \widetilde{\mathcal{W}}'_{m+1} = \mathcal{W}$ of the same type as $\mathcal{W}_0 = \widetilde{\mathcal{W}}_0, \widetilde{\mathcal{W}}_1, \dots, \widetilde{\mathcal{W}}_{m+1} = \mathcal{W}_1^{op}$ and whose image by the retraction retr is precisely this last gallery. In order to obtain the first gallery from the last one, one may "rotate", one by one, each chamber in the gallery around the panel which it has in common with the previous chamber in the gallery. This finite sequence of rotations allowing to finally send $\mathcal{W}_1^{op}$ onto $\mathcal{W}$ can be written, in terms of actions of isometries, as a product $u_{\alpha_0} \cdot u_{\alpha_1} \cdot u_{\alpha_2} \cdots u_{\alpha_m}$. This product is uniquely determined by the gallery $\widetilde{\mathcal{W}}_0', \widetilde{\mathcal{W}}_1', \dots, \widetilde{\mathcal{W}}_{m+1}'$, so by $\mathcal{W}$, and it determines $\mathcal{W}$.

2.5.B Obviously, for every $\Psi \subset \Xi_+$, $\Psi \neq \Xi_+$, $\dim U_\Psi < \dim U_0$. Therefore

$$\Big(U_0 \setminus \bigcup U_\Psi\Big) \cap B(e,\delta) \neq \emptyset \quad \forall \delta > 0\,.$$

We choose and fix $u_0 \in \big(U_0 \setminus \cup U_\Psi\big) \cap B(e,\delta)$ for δ sufficiently small. The main properties of u_0 are that $u_0 F_0$ is a maximal flat whose boundary at infinity intersects $\mathcal{F}_0$ in $\mathcal{W}_0$ and that the Hausdorff distance between W_0 and $u_0 W_0$ is small.

Let now $\Delta_0, \Delta_1, \dots, \Delta_{p_0}$ be all possible fundamental sets of roots, corresponding to the Weyl chambers $\triangleleft A_0, \triangleleft A_1, \dots, \triangleleft A_{p_0}$ of A. For each Δ_i we repeat the previous argument and obtain the small unipotent element u_i, $i \in \{1, 2, \dots, p_0\}$. We thus obtain the finite family of small unipotents $\{u_0, u_1, \dots, u_{p_0}\}$, which we shall use in the sequel.

Notation. We consider the sets $\overline{K} = K/K_A$, $\overline{K}_\alpha = K_\alpha/K_A$ and $\overline{K}_\Lambda = K_\Lambda/K_A$. We denote $\pi_A : K \to \overline{K}$. We denote by $d_{\overline{K}}$ the K-invariant metric on $\overline{K}$. Similarly, $d_{\overline{K}_\alpha}$, $d_{\overline{K}_\Lambda}$ are the K_α and K_Λ-invariant metrics on $\overline{K}_\alpha$ and $\overline{K}_\Lambda$ respectively. By $\bar{k}F_0$ and $\bar{k}W_0$, $\bar{k} \in \overline{K}$, we mean kF_0 and kW_0 respectively. In the sequel we shall often identify $\overline{K}$ with the set of Weyl chambers of vertex x_0 by identifying $\bar{k} \in \overline{K}$ with $\bar{k}W_0$. With this identification, the topology induced on $\overline{K}$ by $d_{\overline{K}}$ is equivalent to the topology induced on the set of Weyl chambers of vertex x_0 by the modified Hausdorff metric [G1, §6] with basepoint x_0. Since there is a bijection between the Weyl chambers of vertex x_0 and the chambers in $\partial_\infty X$ [Dr, Lemma 2.16], we may also identify $\overline{K}$ with the set of chambers in $\partial_\infty X$ by identifying $\bar{k} \in \overline{K}$ with $\bar{k}\mathcal{W}_0$.

For g in G, we denote

$$\Omega(g) = \{\overline{k^{op}} \in \overline{K} \mid k^{op}\mathcal{W}_0 \text{ is opposite to } g\mathcal{W}_0 \text{ in } \partial_\infty X\}\,;$$
$$\Omega_n(g) = \Big\{\overline{k^{op}} \in \Omega(g) \mid \text{ the unique maximal flat } F(k^{op}) \text{ asymptotic to } gW_0 \text{ and } k^{op}W_0 \text{ satisfies } d(x_0, F(k^{op})) < n\Big\}\,.$$

Since the Tits metric is lower semicontinuous with respect to the cone topology [BGS, §4.9], $\Omega(g)$ and $\Omega_n(g)$ are open sets in $\overline{K}$.

Lemma 2.5.1. *Let $g \in G$. Then there exist $m_0, m_1, \dots, m_{p_0}$, positive real numbers depending on g such that $\overline{K} = \cup_{i=0}^{p_0} \Omega_{m_i}(g\sigma_i)$.*

Proof. By [T, Lemma 4.2], each chamber $\mathcal{W}$ in $\partial_\infty X$ has at least an opposite chamber in $g\mathcal{F}_0$. This implies $\overline{K} = \cup_{i=0}^{p_0}\Omega(g\sigma_i)$. If we replace $\Omega(g\sigma_i)$ by $\cup_{m\in\mathbb{R}}\Omega_m(g\sigma_i)$, the compacity of $\overline{K}$ allows to conclude. $\square$

3 Logarithmic Flats and Logarithmic Weyl Chambers

3.1 Definitions. Throughout the whole paper, for a subset A in a metric space (X,d) we denote by $N_\delta(A)$ the δ-neighborhood of $A : \{x \in X \mid d(x,A) < \delta\}$.

In [Su, §9], Sullivan has shown the following *logarithm law for geodesics* for finite volume hyperbolic manifolds. Let Γ be a discrete group of isometries of $\mathbb{H}^{n+1}$ so that $\mathcal{V} = \Gamma\backslash\mathbb{H}^{n+1}$ has finite volume. For every $v \in T_x\mathcal{V}$, let ς_v be the geodesic ray of origin x tangent to v. If we consider the unit fibre bundle $S\mathcal{V}$ endowed with the natural finite measure, then for almost every $v \in S\mathcal{V}$

$$\limsup_{t\to\infty} \frac{d(\varsigma_v(t),\varsigma_v(0))}{\log t} = \frac{1}{n}\,.$$

This means that for every $\varepsilon > 0$ a generic geodesic ray wanders in the cusps of $\mathcal{V}$, not farther than $\left(\frac{1}{n}+\varepsilon\right)\log t$ from its origin at a time t sufficiently large, and that for an infinite number of times t the geodesic ray goes out in the cusps at distance at least $\left(\frac{1}{n}-\varepsilon\right)\log t$ from its origin. Obviously the origin $\varsigma_v(0)$ may be replaced by any fixed point x_0. By raising back to $\mathbb{H}^{n+1}$, one obtains that a generic minimizing geodesic ray in $\mathbb{H}^{n+1}$ never goes farther from Γx_0 than $\left(\frac{1}{n}+\varepsilon\right)\log t$ at the time t sufficiently large, and that an infinite number of times it goes at least as far as $\left(\frac{1}{n}-\varepsilon\right)\log t$ from Γx_0. We are interested only in the first property, even without a constant as precise as $1/n$ and we are also interested in the time T starting from which the property is true. So for a fixed non-uniform lattice Γ in $Isom(\mathbb{H}^{n+1})$, a fixed point x_0 in $\mathbb{H}^{n+1}$ and a fixed time T we call *T-logarithmic geodesic ray* a minimizing geodesic ray ς in $\mathbb{H}^{n+1}$ so that for all times $t \geq T$, $d(\varsigma(t),\Gamma x_0) \leq \frac{2}{n}\log t$ and $d(x_0,\varsigma(0)) \leq \log T$. We call *$T$-logarithmic geodesic* a minimizing geodesic ζ in $\mathbb{H}^{n+1}$ so that for all times t with $|t| \geq T$, $d(\zeta(t),\Gamma x_0) \leq \frac{2}{n}\log t$ and $d(x_0,\zeta(0)) \leq \log T$. The second condition in these two definitions means that the time t is almost the same thing as the distance $d(x_0,\varsigma(t))$ or $d(x_0,\zeta(t))$ respectively. The reason for this condition will become clear in the sequel when we shall try to extend these notions to the higher rank case.

In the symmetric space X of rank $r \geq 2$ the geometry changes because of the presence of maximal flats, which are isometric copies of $\mathbb{R}^r$. In many respects the behaviour of the flats and Weyl chambers in X is similar to the one of geodesics and geodesic rays in $\mathbb{H}^n$. It is then natural, once we considered a non-uniform lattice Γ of isometries of X, to define logarithmic maximal flats and logarithmic Weyl chambers with respect to Γ. In the definition we must also take into account the fact that maximal flats and Weyl chambers are not parametrized only by one parameter ("the time") anymore, but by $r \geq 2$ parameters. But we replace the time t by the distance ρ to a fixed point x which is not too far away from the maximal flat or the vertex of the Weyl chamber respectively.

Let then Γ be a non-uniform irreducible lattice in the semisimple group G. Let X_0 be a subspace of X constructed by cutting off a countable family of open horoballs such that $\Gamma \backslash X_0$ is compact ([L, Section 5]). We recall that we have fixed in section 2.5 a point $x_0 \in X$ and its stabilizer K. We construct X_0 big enough to contain $N_{c_0}(\Gamma x_0)$, with c_0 sufficiently large.

Notation. We denote by $B^X(x,R)$ the open ball of center x and radius R in a metric space X and by $S^X(x,R)$ the sphere of center x and radius R.

We consider the constant $M_0 = \max\big(1, 20(r+1)/\vartheta\big)$, where r is the rank of X and ϑ is the constant defined in the beginning of section 3.2. This constant will appear in the definitions below and will play the same role as $2/n$ in the definitions in $\mathbb{H}^{n+1}$.

Definition 3.1.1. *We call R-logarithmic flat (R-l. flat) in X with respect to x a maximal flat F such that*

(1) *For all $\rho \geq R$, $S(x,\rho) \cap F \subset N_{M_0 \log \rho}(X_0)$,*
(2) *$d(x,F) \leq \log R$.*

Condition (2) means that, up to a bounded perturbation, we may suppose the point x is on the maximal flat F, and condition (1) means that all points in F at distance $\rho \geq R$ from x cannot be farther than $M_0 \log \rho$ from X_0.

Definition 3.1.2. *We call affine torus in G with respect to g the submanifold gA of G, where A is the chosen maximal $\mathbb{R}$-torus. We say that the affine torus gA is an R-logarithmic affine torus (we shall write R-l. affine torus) with respect to g if it has the property that for all $\rho \geq R$, $gS^A(e,\rho) \subset N_{M_0 \log \rho}(\Gamma)$.*

Every affine torus gA with respect to g has the property that gAx_0 is a maximal flat through gx_0. If gA is moreover R-logarithmic with respect to g, then gAx_0 is an R-l. flat with respect to gx_0.

Definition 3.1.3. *A Weyl chamber W in X of vertex x' is called an R-logarithmic incidence (R-l.i.) Weyl chamber with respect to x if $d(x,x') \leq \log R$ and if there exist two R-l. flats with respect to x, F, F', verifying $F(\infty) \cap F'(\infty) = W(\infty)$.*

Another way of formulating this is that a logarithmic incidence Weyl chamber W is at bounded Hausdorff distance from $N_{\log R}(F) \cap N_{\log R}(F')$, where F and F' are logarithmic flats.

For a flat F and a point x in X, we denote the projection of x on F by $\mathrm{proj}_F x$.

Definition 3.1.4. *An R-logarithmic flat F with respect to x whose Weyl chambers of vertex $x' = \mathrm{proj}_F x$ are R-logarithmic incidence with respect to x is called an R-logarithmic branching (R-l.b.) flat with respect to x.*

Let $\mathcal{V} = \Gamma \backslash X$, $\mathcal{V}_G = \Gamma \backslash G$ and $\pi : X \to \mathcal{V}$, $\pi_G : G \to \mathcal{V}_G$ the canonical projections. In our arguments we shall use the known fact that the canonical projection $p : \mathcal{V}_G \to \mathcal{V}$ is a principal fiber bundle with group K over $\mathcal{V}$. We denote $\pi_G(g)$ by $\bar{g}$ and $\pi(x)$ by $\bar{x}$. In the sequel the same name will be used to refer to the projection of a certain object in $\mathcal{V}$ or $\mathcal{V}_G$ as for the object itself in X or G, respectively.

Examples. An *R-logarithmic affine torus in $\mathcal{V}_G$* with respect to $\bar{g}$ is the projection $\pi_G(gA) = \bar{g}A$ of an R-logarithmic affine torus in G with respect to g. An *R-logarithmic incidence Weyl chamber* in $\mathcal{V}$ with respect to $\bar{x}$ is the projection of an R-l.i. Weyl chamber in X with respect to x.

3.2 Density. We show that for almost every point $\bar{x}$ in $\mathcal{V}$, almost every Weyl chamber of vertex $\bar{x}$ is l.i. and almost every maximal flat through $\bar{x}$ is l.b. Henceforth, by "full measure subset" we mean the complementary of a zero-measure subset.

The measure of the set $L_R = \{\bar{g} \in \mathcal{V}_G \mid d(\bar{e}, \bar{g}) \geq R\}$ is of order $e^{-\vartheta R}$, where $\vartheta > 0$ is a constant depending on G. We consider the sequence of sets

$$L_n = \left\{ \bar{g} \in \mathcal{V}_G \mid d(\bar{e}, \bar{g}) \geq \frac{r+1}{\vartheta} \log n \right\},$$

with measures of order $1/n^{r+1}$, where r is the rank of G.

In the torus A we consider the Euclidean spheres $S_n = S^A(e,n)$, and in each sphere we consider a metric 1-net $\mathcal{R}_n \subset S_n$, that is a subset of S_n such that $S_n = N_1(\mathcal{R}_n)$ and for every two elements $g, g' \in \mathcal{R}_n$, $d(g,g') \geq 1$. We construct the sequence of sets

$$\mathfrak{L}_n = \bigcup_{g_{i_n} \in \mathcal{R}_n} \left(L_n g_{i_n}^{-1} \cup L_n g_{i_n}^{-1} u_0^{-1} \cup L_n g_{i_n}^{-1} u_1^{-1} \cup \cdots \cup L_n g_{i_n}^{-1} u_{p_0}^{-1} \right),$$

where $\{u_0, u_1, \dots, u_{p_0}\}$ are the small unipotents chosen in section 2.5. We have that $\mu(\mathfrak{L}_n) \le c' \, n^{r-1} \mu(L_n) \le c \, \frac{1}{n^2}$. This implies that $\sum_{n=1}^{\infty} \mu(\mathfrak{L}_n) < \infty$.

The significance of an element $\bar{g}$ being in $\mathfrak{L}_n$ is the following: the sets $\bar{g}\mathcal{R}_n$ or $\bar{g}u_i\mathcal{R}_n$, $i \in \{0, 1, \dots, p_0\}$, have points in the cusp farther than $\frac{r+1}{\vartheta} \log n$ from $\bar{e}$. The same thing is then true for $\bar{g}S_n$ and $\bar{g}u_iS_n$, up to a variation of order 1, or these are projections in $\mathcal{V}_G$ of the spheres of center g and gu_i respectively and radius n in the affine tori gA and gu_iA. In the sequel we show that almost every element in $\mathcal{V}_G$ is contained in at most a finite number of sets $\mathfrak{L}_n$. For this we need the following classical lemma.

Lemma 3.2.1 (Borel-Cantelli). *Let (X, μ) be a probability space and $\{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$ a family of functions such that $S = \sum_{n=1}^{\infty} \int f_n d\mu < \infty$. Then for every $\varkappa > 0$ we have*

$$\mu\left(\left\{x \in X \;\middle|\; \sum_{n=1}^{\infty} f_n(x) \le \varkappa \cdot S\right\}\right) \ge 1 - \frac{1}{\varkappa}.$$

In particular, $\sum_{n=1}^{\infty} f_n(x) < \infty$ for almost every $x \in X$.

Proof. By a straightforward computation we deduce

$$\mu\left(\left\{x \in X \mid \frac{\sum_{n=1}^{\infty} f_n(x)}{S} \ge \varkappa\right\}\right) \le \frac{1}{\varkappa},$$

and the other desired relations follow. □

We denote $s = c \sum_{n=1}^{\infty} \frac{1}{n^2}$. We apply the lemma to $\{1_{\mathfrak{L}_n} \mid n \in \mathbb{N}\}$ and we obtain:

Lemma 3.2.2. (i) *For almost every $\bar{g} \in \mathcal{V}_G$, $\bar{g}A$ and $\bar{g}u_iA$, $i \in \{0, 1, \dots p_0\}$, are l. affine tori with respect to $\bar{g}$ and $\bar{g}u_i$.*

(ii) *For every $R \in \mathbb{R}_+^*$, the set*

$$\mathrm{Log}(R) := \left\{\bar{g} \in \mathcal{V}_G \;\middle|\; \bar{g}A,\ \bar{g}u_iA \text{ are } R\text{-l. affine tori w.r.t. to } \bar{g} \text{ and } \bar{g}u_i,\ i \in \{0, 1, \dots, p_0\}\right\}$$

has measure larger than $1 - \frac{s}{\log R}$.

Proof. The statement (i) is a consequence of (ii). So it suffices to prove (ii). For every $R \in \mathbb{R}_+^*$, by Lemma 3.2.1, the set

$$\mathrm{Log}'(R) = \left\{\bar{g} \in \mathcal{V}_G \;\middle|\; \sum_{n=1}^{\infty} 1_{\mathfrak{L}_n}(\bar{g}) \le \log R\right\}$$

has a measure larger than $1 - \frac{s}{\log R}$. Let $\bar{g} \in \mathrm{Log}'(R)$. Then card $\{n \mid \bar{g} \in \mathfrak{L}_n\} \le \log R$. This means that, for every $n \ge R$, if $\bar{g} \in \mathfrak{L}_n$ then there exists

$m \in \mathbb{N} \cap [0, 2\log R]$ such that

$$\bar{g} \notin \mathfrak{L}_{n+m} \Leftrightarrow \bar{g}S_{n+m} \cup \bigcup_{i=0}^{p_0} \bar{g}u_i S_{n+m} \subset B\big(\bar{e}, M_0' \log(n+m) + 1\big)\,,$$

where $M_0' = (r+1)/\vartheta$.

So

$$\bar{g}S_n \cup \bigcup_{i=0}^{p_0} \bar{g}u_i S_n \subset B\big(\bar{e}, M_0' \log(n + 2\log R) + 2\log R + 1\big) \subset B(\bar{e}, 4M_0' \log n)\,.$$

Then, for every $\rho \geq R$,

$$\bar{g}S_\rho \cup \bigcup_{i=0}^{p_0} \bar{g}u_i S_\rho \subset B(\bar{e}, 5M_0' \log \rho + 1) \subset B\left(\bar{e}, M_0 \log \rho\right)\,.$$

That is, $\bar{g}A$ and $\bar{g}u_iA$ are R-l. affine tori with respect to $\bar{g}$ and $\bar{g}u_i$, $i \in \{0, 1, \dots, p_0\}$. We conclude that the set $\mathrm{Log}(R)$ contains $\mathrm{Log}'(R)$ and therefore has measure larger than $1 - \frac{s}{\log R}$. □

For a generic $\bar{g}$ satisfying property (i) of Lemma 3.2.2, gAx_0 is a logarithmic flat w.r. to gx_0 and the same is true for each gu_iAx_0 w.r. to gu_ix_0. This implies that gAx_0 is a logarithmic branching flat w.r. to gx_0. This and the Fubini theorem give the following consequence.

Corollary 3.2.3. *For almost every x in X, almost every maximal flat through x is an R-l.b. flat and almost every Weyl chamber of vertex x is R-l.i. with respect to x, for some $R \in \mathbb{R}^*$.*

Remark 3.2.4. *In [KM] it is shown that for almost every point x in X, for almost every maximal flat F through x, one has*

$$\limsup_{\rho\to\infty} \frac{\sup\{d(y, \Gamma x_0) \mid y \in F \cap S(x,\rho)\}}{\log \rho} = c\,,$$

where c is an universal constant.

In the sequel we suppose we have chosen our fixed point $x_0 \in X$ in the full measure set described in the corollary above.

3.3 Horizons of images by quasi-isometry.

3.3.A The group of quasi-isometries.

Definition 3.3.1. *Let X and Y be two metric spaces. An (L, c)-quasi-isometric embedding of X into Y is a map $q : X \to Y$ such that*

$$\tfrac{1}{L} d(x,y) - c \leq d\big(q(x), q(y)\big) \leq L d(x,y) + c\,, \quad \forall x, y \in X\,.$$

If there exists an (L,c)-quasi-isometric embedding of Y into X, q^{-1}, such that

$$d\big(q^{-1}(q(x)),x\big) \le c\,, \quad d\big(q(q^{-1}(y)),y\big) \le c\,, \quad \forall x \in X, y \in Y\,,$$

then q is called an (L,c)-quasi-isometry. The map q^{-1} is called the inverse quasi-isometry. Two metric spaces X and Y are called quasi-isometric if there exists a quasi-isometry between them.

Let $q : \Gamma \to \Gamma$ be an (L,c)-quasi-isometry. The group Γ may be considered with some word metric or with the induced metric; by the Lubotzky-Mozes-Raghunathan theorem ([LuMR1,2]) all these metrics are bilipschitz equivalent on Γ. We can also define a quasi-isometry on the orbit Γx_0 with the induced metric by $q(\gamma x_0) = q(\gamma)x_0$ and extend it to a quasi-isometry on X_0. We shall denote all three of them by q.

If we identify quasi-isometries of Γ which are finite distance one from the other, the set of equivalence classes thus obtained forms a group which we denote by $QI(\Gamma)$ and which we call *the group of quasi-isometries of* Γ.

3.3.B Horizons of infinite sets. Let X be a CAT(0)-space and x a fixed point in X. We recall that by $\tilde{\angle}_x(a,b)$ we denote the comparison angle between a and b seen from x (defined in section 2.3.A).

DEFINITION 3.3.2. *Let A be a subset in $X \setminus \{x\}$ and b a point in $X \setminus \{x\}$. We define*

$$\tilde{\angle}_x(b,A) := \inf\left\{\tilde{\angle}_x(b,a) \;\middle|\; a \in A\right\},$$

which we call the comparison angle between the point b and the set A seen from x. If A and B are two subsets of $X \setminus \{x\}$, we define

$$\tilde{\angle}_x^H(A,B) := \max\left\{\sup_{a\in A}\tilde{\angle}_x(a,B), \sup_{b\in B}\tilde{\angle}_x(b,A)\right\},$$

which we call the Hausdorff comparison angle between the sets A and B seen from x.

In an Euclidean plane $\tilde{\angle}_x(b,A)$ represents the infimum of the angles between the segment $[x,b]$ and the rays of origin x intersecting A (see Figure 1). In other words it represents "how wide one must open one's eyes in order to see the point b and a bit of the set A if one is placed in x". In a general CAT(0)-space, $\tilde{\angle}_x(b,A)$ does not have an interpretation as an infimum of angles anymore. But since angles in Euclidean spaces represent more or less ratios between lengths, the value of $\tilde{\angle}_x(b,A)$ indicates that from a metric point of view the points x and b are placed with respect to the set A similarly to points $\bar{x}$ and $\bar{b}$ with respect to a set $\overline{A}$ in an Euclidean planar configuration with $\tilde{\angle}_{\bar{x}}(\bar{b},\overline{A}) = \tilde{\angle}_x(b,A)$.

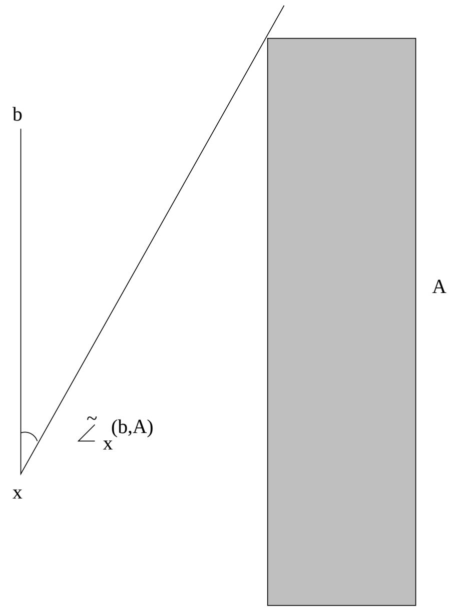

Figure 1

The Hausdorff comparison angle between two sets A and B in an Euclidean plane represents "how wide one must open one's eyes in order to be sure of seeing any of the two sets entirely together with a bit of the other set" (Figure 2). In a CAT(0)-space the information given by $\tilde{\angle}_x^H(A,B)$ is, as previously, only metric; it indicates that from a metric point of view the sets A and B are placed with respect to the point x and to one another as in the planar Euclidean situation described before, with the angle $\tilde{\angle}_x^H(A,B)$.

This interpretation of $\tilde{\angle}_x^H(A,B)$ in a CAT(0)-space, as well as the previous one for $\tilde{\angle}_x(b,A)$, are, of course, only intuitive and not perfectly rigorous, since in most cases the sets A and B cannot be isometrically embedded into an Euclidean plane.

In the sequel we shall be interested mainly in the particular case when the distances to the basepoint x are almost equal and the comparison angles are small. If we have two points a,b such that $d(x,a)/d(x,b)$ is near to 1 and $\tilde{\angle}_x(a,b)$ is very small this means that $d(a,b)$ is very small compared to $d(x,a),d(x,b)$. So, "seen from x", the point a is very close to the point b. Similarly, if we have a point b and a set A such that $d(x,a)/d(x,b)$ is uniformly near to 1, $\forall a\in A$, and $\tilde{\angle}_x(b,A)$ is very small, this means that

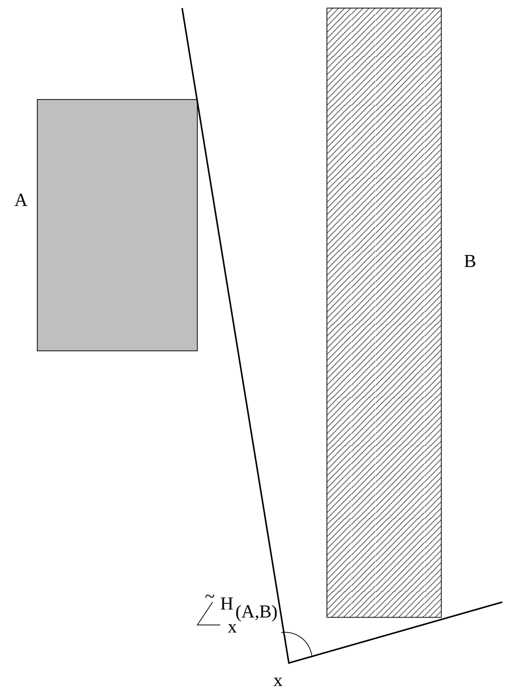

Figure 2

$d(b, A)$ is very small compared to $d(x, A), d(x, b)$, that is "seen from x", the point b is very close to the set A. If two sets A and B have $d(x, a)/d(x, b)$ uniformly near to 1, $\forall a \in A$, $\forall b \in B$, and $\tilde{\angle}_x^H(A, B)$ very small, then the distance of every point in one set to the other set is very small compared to $d(x, A), d(x, B)$, and the two sets are "seen from x" as almost superimposed.

Notation. We denote the closed annulus $\overline{B(x, R)} \setminus B(x, \rho)$ by $\mathfrak{C}(x, \rho, R)$, where $R \geq \rho > 0$. For $\rho > 0$ and $0 < \varepsilon < 1$, we denote $\mathfrak{C}_\varepsilon(x, \rho) := \mathfrak{C}(x, \rho(1 - \varepsilon), \rho(1 + \varepsilon))$.

Definition 3.3.3. *Let A and B be two infinite unbounded subsets of X. We say that the horizon of A is contained in the horizon of B with respect to x if for every $\varepsilon > 0$ there exists R_ε such that for every $\rho \geq R_\varepsilon$ and for $\xi = \varepsilon/100$ we have*

$$\sup \left\{ \tilde{\angle}_x(a, B \cap \mathfrak{C}_\xi(x, \rho)) \;\middle|\; a \in A \cap \mathfrak{C}_\xi(x, \rho) \right\} \leq \varepsilon \,.$$

We say that A and B have the same horizon with respect to $x \in X$ if for every $\varepsilon > 0$ there exists R_ε such that for every $\rho \geq R_\varepsilon$ and for $\xi = \varepsilon/100$

we have

$$\tilde{\angle}_x^H \big(A \cap \mathfrak{C}_\xi(x,\rho), B \cap \mathfrak{C}_\xi(x,\rho)\big) \leq \varepsilon \,.$$

Having the same horizon is an equivalence relation in the family of infinite unbounded subsets of X. We call an equivalence class with respect to this equivalence relation a horizon.

It is not difficult to see that if A and B have the same horizon with respect to x, then they have the same horizon with respect to any point in X. For instance if in Figure 2 we would prolongate the finite strips A and B to infinite strips, they would have the same horizon with respect to x and to any other point.

The significance of the notion of horizon is as follows: the fact that the horizon of A is contained in the horizon of B means that the farther an observer placed at point x looks, the more set A has the tendency to approach set B. Thus in the end, at an infinite distance, A is seen as included in B. Two sets A and B have the same horizon if the farther an observer looks, the more the sets A and B superimpose. So at an infinite distance A and B are seen as coincident.

In the definition of the equivalence relation "same horizon", we prefer to use thin annuli rather than spheres because the sets we will consider in the sequel might not intersect an infinity of spheres of center x.

3.3.C Horizons of images of logarithmic flats In this section we fix an arbitrary point x in X_0, we consider only logarithmic (branching) flats and logarithmic incidence Weyl chambers with respect to x and shall no more mention it each time.

In our reasoning we shall need the following simple lemma about based ultralimits of CAT(0)-spaces. The main ingredient in the proof of this lemma is the convexity of the distance in a CAT(0)-space.

LEMMA 3.3.4. *Let (X,d) be a CAT(0)-space, ω a non-principal ultrafilter and ι_n, η_n two sequences in $\mathbb{R}_+^*$ such that $\lim_\omega \eta_n/\iota_n < 1$. Let $[x,a_n)$ and $[x,b_n)$ be two sequences of non-degenerate geodesic segments or geodesic rays in X such that their limit sets in the ω-based ultralimit $\lim_\omega \big(X, x, \frac{1}{\iota_n} d\big)$ coincide and consist either of a nontrivial geodesic segment or of a geodesic ray. Then the limit sets of $[x,a_n)$ and $[x,b_n)$ in the ω-based ultralimit $\lim_\omega \big(X, x, \frac{1}{\eta_n} d\big)$ coincide.*

Proof. We have two cases: either the common limit set of $[x,a_n)$ and $[x,b_n)$ in $X_\omega = \lim_\omega \big(X, x, \frac{1}{\iota_n} d\big)$ is a nontrivial geodesic segment or it is a ray.

(1) Suppose we are in the second case. Then $\lim_\omega d(x,a_n)/\iota_n = \lim_\omega d(x,b_n)/\iota_n = \infty$. Now let us consider sequences of points on the

segments which give limit points in $\overline{X}_\omega = \lim_\omega \left(X, x, \frac{1}{\eta_n} d\right)$, $t_n = [x, a_n) \cap S(x, \rho\eta_n)$ and $s_n = [x, b_n) \cap S(x, \rho\eta_n)$. The inequality $\lim_\omega \eta_n/\iota_n < 1$ implies that $\eta_n < \iota_n$ $\omega-$almost surely. Then let $T_n = [x, a_n) \cap S(x, \rho\iota_n)$ and $S_n = [x, b_n) \cap S(x, \rho\iota_n)$. By the convexity of the distance in a CAT(0)-space we have that $d(t_n, s_n)/d(x, t_n) \leq d(T_n, S_n)/d(x, T_n)$. That is $d(t_n, s_n)/\rho\eta_n \leq d(T_n, S_n)/\rho\iota_n$. On the other hand, since the limit sets of $[x, a_n)$ and $[x, b_n)$ in X_ω coincide, $\lim_\omega d(T_n, S_n)/\iota_n = 0$. This implies that $\lim_\omega d(t_n, s_n)/\eta_n = 0$. We conclude that the limit sets of $[x, a_n)$ and $[x, b_n)$ in $\overline{X}_\omega$ coincide.

(2) Suppose we are in the first case. Then $\lim_\omega d(x, a_n)/\iota_n = \lim_\omega d(x, b_n)/\iota_n$ is a finite number $\delta > 0$ and $\lim_\omega d(a_n, b_n)/\iota_n = 0$. We then repeat the previous argument in which we replace T_n and S_n by a_n and b_n. □

One can define a natural projection $\pi_0' : \Gamma\backslash X \to \Gamma\backslash X_0$ [L, Section 4], and by means of it one can construct a Γ-invariant projection $\pi_0 : X \to X_0$. This projection might be highly discontinuous, but for our purposes this is not important. We show that the image of a logarithmic flat F by $q \circ \pi_0$ does not completely loose its "flat character". More precisely, we show that $q(\pi_0(F))$ has the same horizon as a fan of vertex $q(x)$ (Proposition 3.3.9). To prove this, we use the fact that the asymptotic cone of X is an Euclidean building and the properties of bilipschitz flats in Euclidean buildings stated in Proposition 2.3.4 and Corollary 2.3.5.

We first show that for every l. flat F, the horizon of $q(\pi_0(F))$ is contained in the horizon of a fan (Lemma 3.3.6). Our first step is to consider an asymptotic cone $\mathbf{K}$ of X with respect to the fixed observation center x. Then $[\pi_0(F)] = [F] = F_\omega$ is an apartment contained in $[X_0]$. The quasi-isometry q induces a bilipschitz map $Q : [X_0] \to [X_0]$ and $[q(\pi_0(F))] = Q(F_\omega)$ is a bilipschitz flat. Then, by Proposition 2.3.4, it locally coincides with an Euclidean cone over a bilipschitz sphere in the building of directions.

Lemma 3.3.5. *Let F be an R-l. flat and $q : X_0 \to X_0$ an (L, c)-quasi-isometry, $q(x) = x$. Suppose that in the asymptotic cone $\mathbf{K} = Con_\omega(X, x, (\iota_n))$ we have*

$$\left[q(\pi_0(F))\right] \cap B(x_\omega, \delta) = \bigcup_{i=0}^{\ell} W_i^\omega \cap B(x_\omega, \delta)\,,$$

where $\delta > 0$, $x_\omega = [x]$ and $W_i^\omega = [W_i^n]$ are Weyl chambers in $\mathbf{K}$ with vertex x_ω. Then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that, if $\xi = \varepsilon/100$, *for ω-almost every n, $\forall \rho \in [R_\varepsilon, \delta\iota_n]$, we have*

$$(\mathbf{In}_\varepsilon(\rho,n))\ \sup\left\{\tilde{\angle}_x\left(z, \bigcup_{i=0}^{\ell} W_i^n \cap S(x,\rho)\right) \,\middle|\, z \in q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,\rho)\right\} \leq \varepsilon\,.$$

The meaning of the conclusion is that the sequence of sets of ℓ Weyl chambers of vertex x, $\{W_1^n, W_2^n, \dots, W_\ell^n\}$, and the sequence of scalars ι_n diverging to ∞ have the property that for every small $\varepsilon > 0$ for ω-almost any n , the set $q(\pi_0(F))$ is seen ε-close to the set $\cup_{i=0}^{\ell} W_i^n$ starting from a uniform distance R_ε and up to $\delta\iota_n$.

Proof of the lemma. STEP 1. First we show that the statement in the Lemma is true only in regular directions. Let $p_x : T_xX \to \Delta_{mod}$ be the projection defined in Definition 2.3.1. Let $\Re \subset Int\,\Delta_{mod}$ be an arbitrary fixed compact set. We prove that for every small $\varepsilon > 0$ there exists $R_\varepsilon(\Re) > 0$ such that ω-a.s. for every $\rho \in [R_\varepsilon(\Re), \delta\iota_n]$ we have

$(\mathbf{In}_\varepsilon(\rho,n,\Re))$

$$\sup\left\{\tilde{\angle}_x\left(z, \bigcup_{i=0}^{\ell} W_i^n \cap S(x,\rho)\right) \,\middle|\, z \in q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,\rho)\,,\ p_x(\overline{xz}) \in \Re\right\} \leq \varepsilon\,.$$

For every $\varepsilon > 0$ and $n \in \mathbb{N}$ we define

$$R_\varepsilon^n(\Re) := \sup\left\{\rho \in (0,\delta\iota_n] \,\middle|\, \mathbf{In}_\varepsilon(\rho,n,\Re) \text{ is not verified}\right\},$$

or $R_\varepsilon^n(\Re) = 1$ if the previous set of ρ's in $(0,\delta\iota_n]$ is empty. We have to prove that $\forall \varepsilon > 0$, $\lim_\omega R_\varepsilon^n(\Re) < +\infty$. We suppose there exists $\varepsilon > 0$ for which $\lim_\omega R_\varepsilon^n(\Re) = +\infty$. For simplicity we denote $R_\varepsilon^n(\Re)$ by R_n. Since for every $\varepsilon_1 \in (0,\varepsilon)$ we have $R_\varepsilon^n(\Re) \leq R_{\varepsilon_1}^n(\Re)$, we may suppose ε as small as we want. We take $\varepsilon = d(\Re, \partial\Delta_{mod})/100$.

For ω-almost every n in $\mathbb{N}$ there exists $z_n \in q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,R_n)$ satisfying

$$p_x(\overline{xz_n}) \in \Re \text{ and } \tilde{\angle}_x\left(z_n, \bigcup_{i=0}^{\ell} W_i^n \cap S(x,R_n)\right) \geq \varepsilon\,. \tag{3.1}$$

Obviously $\lim_\omega R_n/\iota_n \leq \delta$. We show that $\lim_\omega R_n/\iota_n = 0$. We suppose on the contrary that $\lim_\omega R_n/\iota_n = \delta' > 0$. In the asymptotic cone $\mathbf{K} = Con_\omega(X,x,(\iota_n))$ the limit point $z_\omega = [z_n]$ is a point in $[q(\pi_0(F))] \cap \mathfrak{C}_\xi(x_\omega,\delta')$ with the property that $\tilde{\angle}_{x_\omega}\big(z_\omega, \cup_{i=0}^{\ell} W_i^\omega \cap S(x,\delta')\big) \geq \varepsilon$. By eventually displacing z_ω at a distance at most $\xi\cdot\delta'$ from its initial position and diminishing ε to $\varepsilon-\xi$ in the previous inequality, we may suppose z_ω is in $S(x_\omega,\delta')$. Then z_ω is in $[q(\pi_0(F))] \cap \overline{B(x_\omega,\delta)}$.

On the other hand, the hypothesis of the lemma can be easily extended to closed balls, so we have

$$\left[q(\pi_0(F))\right] \cap \overline{B(x_\omega, \delta)} = \cup_{i=0}^{\ell} W_i^\omega \cap \overline{B(x_\omega, \delta)}\,.$$

This means that the point z_ω must coincide with a point in the set $\cup_{i=0}^{\ell} W_i^\omega \cap S(x, \delta')$, so it cannot make a strictly positive comparison angle with the same set. We have obtained a contradiction.

Thus $\lim_\omega R_n/\iota_n = 0$ which is equivalent to $\lim_\omega \iota_n/R_n = +\infty$.

We consider the cone $\overline{\mathbf{K}} = Con_\omega(X, x, (R_n))$. The limit set $[q(\pi_0(F))] = \overline{Q}(\overline{F}_\omega)$ is a bilipschitz flat and $\overline{W}_i^\omega = [W_i^n \cap B(x, \delta\iota_n)]$ is a Weyl chamber.

Let $\Sigma_{x_\omega}\overline{\mathbf{K}}$ be the building of directions in $x_\omega = [x]$ and $p_{x_\omega} : \Sigma_{x_\omega}\overline{\mathbf{K}} \to \Delta_{mod}$ the projection defined in section 2.3.B. The point $z_\omega = [z_n] \in \overline{Q}(\overline{F}_\omega) \cap \mathfrak{C}_\xi(x_\omega, 1)$ has the properties that $p_{x_\omega}(\overline{x_\omega z_\omega}) \in \Re$ and

$$\tilde{\angle}_{x_\omega}(z_\omega, y_\omega) \geq \varepsilon,\ \forall y_\omega \in \bigcup_{i=0}^{\ell} \overline{W}_i^\omega \cap S(x_\omega, 1)\,. \tag{3.2}$$

On the other hand, by the maximality of R_n we have that

$$\begin{aligned} &\forall\, u_\omega \in \overline{Q}(\overline{F}_\omega) \cap \mathfrak{C}_\xi(x_\omega, \rho)\,,\ \rho > 1\,,\quad \text{with } p_{x_\omega}(\overline{x_\omega u_\omega}) \in \Re\,,\\ &\exists\, v_\omega \in \bigcup_{i=0}^{\ell} \overline{W}_i^\omega \cap S(x_\omega, \rho) \text{ such that } \tilde{\angle}_{x_\omega}(u_\omega, v_\omega) \leq \varepsilon\,. \end{aligned} \tag{3.3}$$

Proposition 2.3.4 (iii), implies that the geodesic $[x_\omega, z_\omega]$ can be extended to a geodesic ray $[x_\omega, \beta)$ with $[z_\omega, \beta) \subset \overline{Q}(\overline{F}_\omega)$. Since $p_{x_\omega}(\overline{x_\omega z_\omega}) \in \Re$, we have $P(\beta) \in \Re$.

By (3.3), for every $z_k = [z_\omega, \beta) \cap S(x_\omega, k)$, with $k \in \mathbb{N}$ sufficiently large, there exists $v_k \in \cup_{i=0}^{\ell} \overline{W}_i^\omega \cap S\,(x_\omega, k)$ such that $\tilde{\angle}_{x_\omega}(v_k, z_k) \leq \varepsilon$. Let γ be the point in $\cup_{i=0}^{\ell} \overline{W}_i^\omega(\infty)$ which appears as the limit of (v_k) or of a subsequence of it. Using the semicontinuity of the comparison angle [KlL, Lemma 2.3.1] we obtain that the Tits distance between β and γ is at most ε. The fact that $P(\beta) \in \Re$ and the choice of ε as $d(\Re, \partial\Delta_{mod})/100$ imply that β and γ are in the interior of the same chamber. Since $\gamma \in \cup_{i=0}^{\ell} \overline{W}_i^\omega(\infty)$, β and γ must be both in one of these chambers, $\overline{W}_{i_0}^\omega(\infty)$. Then $[x_\omega, \beta) \subset \overline{W}_{i_0}^\omega$, so $z_\omega \in \overline{W}_{i_0}^\omega$. The last relation contradicts the inequality (3.2).

Step 2. For every $\varepsilon > 0$ and $n \in \mathbb{N}$ we define

$$R_\varepsilon^n := \sup\left\{\rho \in (0, \delta\iota_n] \mid \mathbf{In}_\varepsilon(\rho, n) \text{ is not verified}\right\},$$

or $R_\varepsilon^n = 1$ if the previous set of ρ's in $(0, \delta\iota_n]$ is empty. We show $\lim_\omega R_\varepsilon^n < +\infty$. Suppose that for some $\varepsilon > 0$, $\lim_\omega R_\varepsilon^n = +\infty$. We denote R_ε^n

by R_n. Take $\overline{\mathbf{K}} = Con_\omega(X, x, (R_n))$. We obtain the same kind of limit sets and of relations in $\overline{\mathbf{K}}$ as in Step 1. The only difference is that the point $z_\omega \in \overline{Q}(\overline{F}_\omega) \cap \mathfrak{C}_\xi(x_\omega, 1)$ does not have the property that $p_{x_\omega}(\overline{x_\omega z_\omega})$ is in some compact $\Re \subset Int\,\Delta_{mod}$ anymore. But z_ω satisfies the inequality (3.2). By Step 1, every u in $\overline{Q}(\overline{F}_\omega)$ with $p_{x_\omega}(\overline{x_\omega u}) \in Int\ \Delta_{mod}$ is contained in $\cup_{i=0}^{\ell} \overline{W}_i^\omega$. This and Corollary 2.3.5 imply that $\overline{Q}(\overline{F}_\omega) \subset \cup_{i=0}^{\ell} \overline{W}_i^\omega$. In particular z_ω is contained in $\cup_{i=0}^{\ell} \overline{W}_i^\omega$, which contradicts the inequality (3.2). □

LEMMA 3.3.6. *Let F be an R-l. flat and $q : X_0 \to X_0$ an (L, c)-quasi-isometry, $q(x) = x$. There exists a fan of vertex x, $\cup_{j=0}^{p} W_j$, with p bounded by a constant $\wp$ depending only on L, such that the horizon of $q(\pi_0(F))$ is contained in the horizon of $\cup_{j=0}^{p} W_j$ with respect to x.*

Proof. We start with the results provided by Lemma 3.3.5. In the symmetric space X, let W_i be the ω-limit of the sequence of sets $(W_i^n \cap B(x, \delta\iota_n))$ in the modified Hausdorff metric [G1, §6]. It is easy to see that the ω-based ultralimit of the sequence $(\cup_{i=0}^{\ell}(W_i^n \cap B(x, \delta\iota_n)), d, x)$ is isometric with $(\cup_{i=0}^{\ell} W_i, d, x)$. This remark and Lemma 3.3.4 imply that the Weyl chambers $\{W_i \mid i \in \{0, 1, 2, \dots \ell\}\}$ have the same adjacencies as the truncated Weyl chambers $\{W_i^\omega \cap B(x_\omega, \delta) \mid i \in \{0, 1, 2, \dots, \ell\}\}$ in $Con_\omega(X, x, (\iota_n))$, and there might be even some extra adjacencies, and that $(\cup_{i=0}^{\ell} W_i(\infty), d_T)$ is a lipschitz sphere.

We fix a small constant $\lambda > 0$ and we denote by

$$R_n := \sup\left\{\rho \in (0, \delta\iota_n] \big| W_i^n \cap B(x, \rho) \subset N_\lambda(W_i)\,,\ \forall i \in \{0, 1, 2, \dots, \ell\}\right\}.$$

Since W_i is the ω-limit of $(W_i^n \cap B(x, \delta\iota_n))$ in the modified Hausdorff metric, we have that $\lim_\omega R_n = \infty$. Let $\mathbf{K} = Con_\omega(X, x, (\sqrt{R_n}))$. We show that in this cone we have

$$\big[q(\pi_0(F))\big] = Q(F_\omega) \subset \bigcup_{i=0}^{\ell} [W_i^n]\,. \tag{3.4}$$

Let z_ω be an arbitrary point of $Q(F_\omega)$. Suppose z_ω is at distance η from x. Then z_ω is a limit point of a sequence of points $z_n \in q(\pi_0(F)) \cap \mathfrak{C}_{\xi_n}(x, \eta \cdot \sqrt{R_n})$, where $\lim_\omega \xi_n = 0$. For every $\varepsilon > 0$ if n is big enough $\eta \cdot \sqrt{R_n}$ is bigger that R_ε given by Lemma 3.3.5. Thus for ω -almost every n we have that

$$\tilde{\angle}_x\bigg(z_n, \bigcup_{i=0}^{\ell} W_i^n \cap S\left(x, \eta \cdot \sqrt{R_n}\right)\bigg) \le \varepsilon\,.$$

In the asymptotic cone $\mathbf{K}$ this gives that

$$\tilde{\angle}_{x_\omega}\left(z_\omega, \bigcup_{i=0}^{\ell}[W_i^n] \cap S(x,\eta)\right) \le \varepsilon\,.$$

Since this is true for every $\varepsilon > 0$ we may conclude that $z_\omega \in \cup_{i=0}^{\ell}[W_i^n] \cap S(x,\eta)$. Thus we have shown the inclusion (3.4).

We also have that $\cup_{i=0}^{\ell}[W_i^n] \subset \cup_{i=0}^{\ell}[W_i] = \cup_{i=0}^{\ell} W_i^\omega$, and $\cup_{i=0}^{\ell} W_i^\omega$ is an Euclidean cone over $(\cup_{i=0}^{\ell} W_i^\omega(\infty), d_T)$, which is isometric to $(\cup_{i=0}^{\ell} W_i(\infty), d_T)$. Here d_T denotes the Tits metric on the boundary at infinity. For $\delta_1 > 0$ sufficiently small, $Q(F_\omega) \cap B(x_\omega, \delta_1) = \cup_{j=0}^{p} \widetilde{W_j} \cap B(x_\omega, \delta_1)$, where $\cup_{j=0}^{p} \widetilde{W_j} \cap B(x_\omega, \delta_1)$ is a δ_1-Euclidean cone over a bilipschitz sphere in $\Sigma_{x_\omega}\mathbf{K}$. Thus we have $\cup_{j=0}^{p} \widetilde{W_j} \cap B(x_\omega, \delta_1) \subset \cup_{i=0}^{\ell} W_i^\omega \cap B(x_\omega, \delta_1)$, so $\cup_{j=0}^{p} (\widetilde{W_j})_{x_\omega} \subset \cup_{i=0}^{\ell} (W_i^\omega)_{x_\omega}$ in $\Sigma_{x_\omega}\mathbf{K}$. This implies that $\cup_{i=0}^{\ell} (W_i^\omega)_{x_\omega}$ contains a bilipschitz sphere and, as it is isometric to $\cup_{i=0}^{\ell} W_i^\omega(\infty)$, we have the same thing for $\cup_{i=0}^{\ell} W_i^\omega(\infty)$. Let $\cup_{j=0}^{p} W_j^\omega(\infty) \subset \cup_{i=0}^{\ell} W_i^\omega(\infty)$ be this bilipschitz sphere. We then have $Q(F_\omega) \cap B(x_\omega, \delta_1) = \cup_{j=0}^{p} W_j^\omega \cap B(x_\omega, \delta_1)$. We apply again Lemma 3.3.5 and we conclude that for every $\varepsilon > 0$ there exists R_ε such that ω-almost surely $\forall \rho \in [R_\varepsilon, \delta_1\sqrt{R_n}]$,

$$\sup\left\{\tilde{\angle}_x\left(z, \bigcup_{j=0}^{p} W_j \cap S(x,\rho)\right) \,\middle|\, z \in q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,\rho)\right\} \le \varepsilon\,. \qquad (3.5)$$

Thus for $\varepsilon > 0$ arbitrary fixed and the corresponding R_ε the statement is true for an infinite subsequence of $\sqrt{R_n}$ diverging to ∞. So in the end we obtain that for every $\rho \in [R_\varepsilon, \infty)$ the inequality (3.5) is verified. But this means precisely that the horizon of $q(\pi_0(F))$ is contained in the horizon of $\cup_{j=0}^{p} W_j$ with respect to x. □

We now prove an uniformity result which shall be needed in section 4.2 to prove our main uniformity results Propositions 4.2.6 and 4.2.7. This result says mainly that the distance R_ε starting from which the image of an R-logarithmic flat (with R fixed) is "seen ε-close to a fan" doesn't depend on the flat nor on the quasi-isometry.

Lemma 3.3.7 (uniformity lemma). *For every $\varepsilon > 0$, $R > 0$, $L \ge 1$ and $c > 0$ there exists $R_\varepsilon = R_\varepsilon(R, L, c)$ such that if F is an R-l. flat, $q : X_0 \to X_0$ is an (L,c)-quasi-isometry, $q(x) = x$, and the horizon of $q(\pi_0(F))$ is contained in the horizon of a fan $\cup_{j=0}^{p} W_j$ with respect to x,*

$p \leq \wp$, then for every $\rho \geq R_\varepsilon$ and for $\xi = \varepsilon/100$, we have

$$\sup\left\{ \tilde{\angle}_x\left(z, \bigcup_{j=0}^{p} W_j \cap S(x,\rho) \right) \,\middle|\, z \in q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,\rho) \right\} \leq \varepsilon\,.$$

Proof. Step 1. We first show that the statement is true in regular directions. Let $\varepsilon > 0$, $R > 0$, $L \geq 1$, $c > 0$ and $\Re$ compact set contained in $Int\,\Delta_{mod}$. We show there exists R_ε such that if F, q and $\cup_{j=0}^{p} W_j$ are as before, then $\forall \rho \geq R_\varepsilon$ we have

$$\sup\left\{ \tilde{\angle}_x\left(z, \bigcup_{j=0}^{p} W_j \cap S(x,\rho) \right) \,\middle|\, z \in q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,\rho),\, p_x(\overline{xz}) \in \Re \right\} \leq \varepsilon\,.$$

We argue by contradiction and suppose that for certain $\varepsilon > 0$, $R > 0$, $L \geq 1$, $c > 0$ and $\Re \subset Int\,\Delta_{mod}$, there exist sequences (F_n), $q_n : X_0 \to X_0$, $q_n(x) = x$, and $\cup_{j=0}^{p_n} W_j^n$, $p_n \leq \wp$, such that if we denote

$$R_n^\varepsilon = \sup\left\{ \rho > 0 \,\middle|\, \exists z \in q_n(\pi_0(F_n)) \cap \mathfrak{C}_\xi(x,\rho),\, p_x(\overline{xz}) \in \Re\,, \right.$$
$$\left. \tilde{\angle}_x\left(z, \bigcup_{j=0}^{p_n} W_j^n \cap S(x,\rho) \right) \geq \varepsilon \right\},$$

then $R_n^\varepsilon \to \infty$.

Since $\varepsilon_1 \leq \varepsilon$ implies $R_n^{\varepsilon_1} \geq R_n^\varepsilon$, we may suppose $\varepsilon \leq d(\Re, \partial\Delta_{mod})/100$. For simplicity we denote R_n^ε by R_n. There exists a sequence $z_n \in q_n(\pi_0(F_n)) \cap \mathfrak{C}_\xi(x, R_n)$ such that $p_x(\overline{xz_n}) \in \Re$ and $\tilde{\angle}_x(z_n, \cup_{j=0}^{p_n} W_j^n \cap S(x,R_n)) \geq \varepsilon$. In $Con_\omega(X, x, (R_n))$ we obtain the same situation as in the Step 1 of the proof of Lemma 3.3.5, with $\overline{Q}(\overline{F}_\omega) = [q_n(\pi_0(F_n))]$ and $\cup_{i=0}^{\ell} \overline{W}_i^\omega = [\cup_{j=0}^{p_n} W_j^n]$ (since $p_n \leq \wp$ there are only a finite number of possible configurations for $\cup_{j=0}^{p_n} W_j^n(\infty)$ and ω chooses one of them). The sequence (z_n) has a limit point $z_\omega \in \overline{Q}(\overline{F}_\omega) \cap \mathfrak{C}_\xi(x,1)$ such that $p_{x_\omega}(\overline{x_\omega z_\omega}) \in \Re$ and $\tilde{\angle}_{x_\omega}(z_\omega, \cup_{i=0}^{\ell} \overline{W}_i^\omega \cap S(x,1)) \geq \varepsilon$. On the other hand, by the maximality of R_n, the points of $\overline{Q}(\overline{F}_\omega)$ which are in the regular directions $\Re$ from x_ω and on farther annuli than z_ω are "seen as ε-close" to the corresponding spheres on $\cup_{i=0}^{\ell} \overline{W}_i^\omega$. This property can be expressed formally exactly as in (3.3). With an argument identical to the one in Step 1 of the proof of Lemma 3.3.5, we obtain a contradiction.

Step 2. We derive the conclusion of the lemma from Step 1 by an argument analogous to the one in Step 2 of the proof of Lemma 3.3.5. $\square$

Among other important consequences, the previous lemma allows to say that in any asymptotic cone the limit set of a sequence of quasi-isometric images of R-logarithmic flats, $[q_n(\pi_0(F_n))]$, is contained in a fan, provided all quasi-isometries have the same constants. This is formulated in the next corollary.

Corollary 3.3.8. *Let $L > 1$, $c > 0$ and $R \geq 0$ be fixed constants. Let (F_n) be a sequence of R-logarithmic flats with respect to a sequence of points $x_n \in X_0$, and let $q_n : X_0 \to X_0$ be a sequence of (L,c)-quasi-isometries, $q_n(x_n) = x_n$. For each $n \in \mathbb{N}$, the horizon of $q_n(\pi_0(F_n))$ is contained in the horizon of a fan $\cup_{j=0}^{p_n} W_n^j$ with respect to x_n, $p_n \leq \wp$.*

Then in every asymptotic cone $Con_\omega(X,(x_n),(\iota_n))$, the bilipschitz flat $[q_n(\pi_0(F_n))]$ is contained in the union of Weyl chambers $[\cup_{j=0}^{p_n} W_j^n]$.

Proof. We consider an asymptotic cone $Con_\omega(X,(x_n),(\iota_n))$. The limit set of the sequence of fans is a union of Weyl chambers $\cup_{i=0}^{\ell} \overline{W}_i^\omega = \left[\cup_{j=0}^{p_n} W_j^n\right]$. We shall show that this fan contains the bilipschitz flat $[q_n(\pi_0(F_n))]$.

We fix $\varepsilon > 0$ an arbitrary small number. Let R_ε be the distance depending on the given ε, R, L, c such that the conclusion of the Lemma 3.3.7 holds. Let z_ω be a point in the bilipschitz flat $[q_n(\pi_0(F_n))]$ which is at distance $\delta > 0$ from $x_\omega = [x_n]$. Then $z_\omega = [z_n]$, where z_n is a point in $q_n(\pi_0(F_n))$ which is at a distance ρ_n from x_n, $\lim_\omega \rho_n/\iota_n = \delta$. In particular z_n is almost surely at distance at least R_ε from x_n. Then, by Lemma 3.3.7, $\tilde{\angle}_{x_n}(z_n, \cup_{j=0}^{p_n} W_j^n \cap S(x,\rho_n)) \leq \varepsilon$. In the asymptotic cone this gives $\tilde{\angle}_{x_\omega}(z_\omega, \cup_{i=0}^{\ell} \overline{W}_i^\omega \cap S(x,\delta)) \leq \varepsilon$. Since this is true for every $\varepsilon > 0$ we finally obtain that z_ω is contained in $\cup_{i=0}^{\ell} \overline{W}_i^\omega \cap S(x,\delta)$. □

Proposition 3.3.9. *For every l. flat F and every (L,c)-quasi-isometry $q : X_0 \to X_0$, $q(x) = x$, there is a fan of vertex x, $\cup_{j=0}^{p} W_j$, such that $q(\pi_0(F))$ has the same horizon as $\cup_{j=0}^{p} W_j$ with respect to x.*

Proof. For F and q fixed, we consider the fan given by Lemma 3.3.6. We argue by contradiction and assume there exists $\varepsilon > 0$, a sequence $R_n \to \infty$ and a sequence $z_n \in \cup_{j=0}^{p} W_j \cap \mathfrak{C}_\xi(x,R_n)$ such that ω-almost surely

$$\tilde{\angle}_x\big(z_n, q(\pi_0(F)) \cap \mathfrak{C}_\xi(x,R_n)\big) > \varepsilon\,.$$

In the cone $Con_\omega(X,x,(R_n))$, we have the inclusion $Q(F_\omega) \subseteq \cup_{j=0}^{p} W_j^\omega$ and both sets are bilipschitz flats. This implies that the two sets coincide. On the other hand, there exists $z_\omega = [z_n] \in \cup_{j=0}^{p} W_j^\omega \cap \mathfrak{C}_\xi(x_\omega,1)$ such that $\tilde{\angle}_{x_\omega}(z_\omega, Q(F_\omega) \cap \mathfrak{C}_\xi(x_\omega,1)) \geq \varepsilon$. We thus obtain a contradiction. □

Corollary 3.3.10. *For every l.i. Weyl chamber W and every (L,c)-quasi-isometry $q : X_0 \to X_0$, $q(x) = x$, there exists a finite union of Weyl* *chambers of vertex* x, $\cup_{l=0}^{s} W_l$, *such that* $q(\pi_0(W))$ *has the same horizon as* $\cup_{l=0}^{s} W_l$ *with respect to* x.

Proof. Let W be an l.i. Weyl chamber. There exist F, F' l. flats satisfying $F(\infty) \cap F'(\infty) = W(\infty)$. Corresponding to F and F', by Proposition 3.3.9, we have two fans of vertex x, $\cup_{j=0}^{p} W'_j$ and $\cup_{k=0}^{m} W''_k$. Let $\cup_{l=0}^{s} W_l = (\cup_{j=0}^{p} W'_j) \cap (\cup_{k=0}^{m} W''_k)$. We show that $q(\pi_0(W))$ has the same horizon as $\cup_{l=0}^{s} W_l$ with respect to x.

Suppose, on the contrary, that there exists $\varepsilon > 0$, a sequence $R_n \to \infty$ and a sequence $z_n \in A_n = q(\pi_0(W)) \cap \mathfrak{C}_\xi(x, R_n)$ or $z_n \in B_n = \cup_{l=0}^{s} W_l \cap \mathfrak{C}_\xi(x, R_n)$ such that $\tilde{\angle}_x(z_n, B_n) \geq \varepsilon$ or $\tilde{\angle}_x(z_n, A_n) \geq \varepsilon$, respectively. In the cone $Con_\omega(X, x, (R_n))$, $F_\omega = [F] = [\pi_0(F)]$ and $F'_\omega = [F'] = [\pi_0(F')]$ are maximal flats (apartments) contained in $[X_0]$ and $W_\omega = [W]$ is their intersection. Let $Q : [X_0] \to [X_0]$ be the bilipschitz map induced by the quasi-isometry q. Then we have the sequence of equalities: $Q(W_\omega) = Q(F_\omega) \cap Q(F'_\omega) = [\cup_{j=0}^{p} W'_j] \cap [\cup_{k=0}^{m} W''_k] = \cup_{j=0}^{p} (W'_j)_\omega \cap \cup_{k=0}^{m} (W''_k)_\omega = \cup_{l=0}^{s} (W_l)_\omega$. On the other hand the limit point z_ω of the sequence (z_n) is in one of the two sets $Q(W_\omega) \cap \mathfrak{C}_\xi(x, 1)$ and $\cup_{l=0}^{s} (W_l)_\omega \cap \mathfrak{C}_\xi(x, 1)$ and it forms a comparison angle which is at least ε with the other set. This contradicts the fact that the two sets $Q(W_\omega)$ and $\cup_{l=0}^{s} (W_l)_\omega$ coincide. □

We shall sometimes say that the quasi-isometry q *associates* to the l. flat F (to the l.i. Weyl chamber W) the fan $\cup_{j=0}^{p} W_j$ (the finite union of Weyl chambers $\cup_{l=0}^{s} W_l$).

REMARK 3.3.11. (1) *We may replace the assumptions* $q(x) = x$ *and* $q_n(x_n) = x_n$ *in Corollary 3.3.8 by* $q(x) = y$ *with* y *an arbitrary point in* X_0 *and* $q_n(x_n) = y_n$ *with* (y_n) *an arbitrary sequence of points in* X_0, *respectively. In this case, the fans* $\cup_{j=0}^{p} W_j$ *have* y *as a basepoint. This follows from the fact that we may compose* q *to the left with elements from* Γ *and we may move the basepoints a bounded distance away, without changing the results.*

(2) *If two l.i. Weyl chambers are asymptotic, the unions of Weyl chambers associated to them by* q *are asymptotic.*

4 One-to-one Mapping between Maximal Flats

4.1 The chosen set of logarithmic branching flats We choose a dense subset of l.b. flats with the property that their points with respect to which they are l.b. form a very big subset within the flat.

Let $c_0 > 0$ be a constant such that $N_{c_0}(\Gamma x_0) \subset X_0$. Let $m \in \mathbb{N}$. The set $CL_m := \{\bar{g} \in \mathcal{V}_G \mid d(\bar{e}, \bar{g}) \leq c_0 m\}$ has a measure larger than $1 - e^{-\vartheta c_0 m}$, and the set $\operatorname{Log}(R_0 m)$ has a measure larger than $1 - \frac{s}{\log(R_0 m)}$ (by Lemma 3.2.2 (ii)). Their intersection

$$\mathcal{S}(m) = CL_m \cap \operatorname{Log}(R_0 m)$$

has a measure larger than $1 - \varepsilon_m$, where $\varepsilon_m \to 0$ as $m \to \infty$. We choose c_0 and R_0 sufficiently large so that ε_1 should already be very small.

We fix $m \in \mathbb{N}$. In the sequel we will denote, as in paragraph 2.4.A, the Lebesgue measure of a finite dimensional Euclidean space E by ν or, sometimes, by ν_E, to avoid confusion. The action of the $\mathbb{R}$-torus A on $\mathcal{V}_G$ is ergodic ([M], [Zi]). By applying Birkhoff's theorem to the action of A on $\mathcal{V}_G$ and to the function $1_{\mathcal{S}(m)}$, we conclude that for almost every $\bar{g} \in \mathcal{V}_G$

$$\lim_{\rho \to \infty} \frac{1}{\nu(B^A(e,\rho))} \int_{B^A(e,\rho)} 1_{\mathcal{S}(m)}(\bar{g}a) d\nu(a) = \mu(\mathcal{S}(m)) > 1 - \varepsilon_m \,.$$

Equivalently, for almost every $\bar{g} \in \mathcal{V}_G$, for ρ sufficiently large we have

$$\nu\big(\big\{a \in B^A(e,\rho) \mid \bar{g}a \in \mathcal{S}(m)\big\}\big) > (1 - \varepsilon_m)\nu(B^A(e,\rho)) \,.$$

That is,

$$\nu\big(\big\{a \in B^A(e,\rho) \bigm| d(\bar{e}, \overline{ga}) \leq c_0 m \text{ and } gaA\,,\ gau_iA \text{ are } R_0 m\text{-l. with} \\ \text{respect to } ga \text{ and } gau_i\,,\ i \in \{0,1,\dots,p_0\}\big\}\big) > (1 - \varepsilon_m)\nu(B^A(e,\rho)) \,. \quad (4.1)$$

For a maximal flat F in the symmetric space X we denote

$$\mathcal{S}_F(m) = \big\{y \in F \bigm| d(\bar{y}, \bar{x}_0) \leq c_0 m \text{ and } F \text{ is an } R_0 m\text{-l.b. flat w.r. to } y\big\} \,.$$

By (4.1), we obtain that for almost every g in G, in the flat $F = gF_0$ through $x = gx_0$ we have, for ρ big enough,

$$\nu\big(B^F(x,\rho) \cap \mathcal{S}_F(m)\big) > (1 - \varepsilon_m)\nu(B^F(x,\rho)) \,.$$

We fix a set of roots of A, Λ, take the action of A_Λ on $\mathcal{V}_G$ and do the same reasoning as before. By repeating the same argument for all subsets Λ of Ξ and taking intersections of full measure subsets, we conclude that for almost every g in G, the flat $F = gF_0$ through the point $x = gx_0$ has the following property:

$(P_x(m))$ If $\{H_\Lambda\}_{\Lambda \subset \Xi}$ is the set of all its singular planes through x, for every $H \in \{F\} \cup \{H_\Lambda \mid \Lambda \subset \Xi\}$, we have, for ρ sufficiently large,

$$\nu_H\big(B^H(x,\rho) \cap \mathcal{S}_F(m)\big) > (1 - \varepsilon_m)\nu_H(B^H(x,\rho)) \,.$$

The significance of this property is that the set of points of the flat F contained in the c_0m-neighborhood of the orbit Γx_0 and with respect to which F is an R_0m-l.b. flat is not only a big set in F but also it intersects any singular (hyper)plane of F through x in a big subset.

Now we let m vary in $\mathbb{N}$. Since a countable intersection of full measure subsets is a full measure subset, we may conclude that for almost every g in G, the flat $F = gF_0$ possesses all the sequence of properties $(P_{gx_0}(m))$, $m \in \mathbb{N}$.

DEFINITION 4.1.1. *We call good logarithmic branching flat (good l.b. flat) with respect to $x \in X$ an l.b. flat with respect to x, containing x and possessing all the sequence of properties $(P_x(m))$, $m \in \mathbb{N}$.*

4.2 One-to-one mapping between maximal flats. From now on we also need the hypothesis that the group G does not have rank one factors. In this section we fix again a point x in X_0. Let F be a good l.b. flat with respect to x. We show that the fan associated to F by q is a fan over an apartment. We do this by going into an asymptotic cone. The limit image F_ω of F is contained in the limit image $[X_0]$ of X_0. Moreover, in almost every point y of it, F_ω branches in $[X_0]$, i.e. all Weyl chambers of F_ω of vertex y appear as intersections of F_ω with another maximal flat contained in $[X_0]$. This will force $Q(F_\omega)$ to be a maximal flat F'_ω.

PROPOSITION 4.2.1. *For every good l.b. flat F with respect to x and every (L,c)-quasi-isometry $q : X_0 \to X_0$, $q(x) = x$, there exists a fan over an apartment, $\cup_{i=0}^{p_0} W_i$, of vertex x, having the same horizon as $q(\pi_0(F))$ with respect to x.*

Moreover, to each Weyl chamber $\widetilde{W}_j$ in F of vertex x corresponds a unique Weyl chamber W_{i_j}, $i_j \in \{0, 1, \dots, p_0\}$, having the same horizon as $q(\pi_0(\widetilde{W}_j))$ with respect to x. The bijection $\widetilde{W}_j(\infty) \to W_{i_j}(\infty)$ induces a simplicial isomorphism between the two apartments $F(\infty)$ and $\cup_{i=0}^{p_0} W_i(\infty)$.

Proof. We first prove that the fan $\cup_{i=0}^{p} W_i$ associated to F by q is a fan over an apartment. Let $\mathbf{K} = Con_\omega(X, x, (\iota_n))$ be an asymptotic cone. Since X has no rank one factors, $\mathbf{K}$ is an Euclidean building with no rank one factors. Let $\mathbf{K} = \prod_{j=1}^{\varkappa} \mathbf{K}_j$, $\varkappa \in \mathbb{N}$, be the decomposition of $\mathbf{K}$ into irreducible factors, where rank $\mathbf{K}_j = r_j \geq 2$, $\forall j \in \{1, 2, \dots, \varkappa\}$. Suppose we have written the factors in the increasing order of their ranks. For every $J \subset \{1, 2, \dots, \varkappa\}$ we define the projection $\pi_J : \mathbf{K} \to \prod_{j=1, j \notin J}^{\varkappa} \mathbf{K}_j$.

Let $F_\omega = [F] = [\pi_0(F)]$, $W_i^\omega = [W_i]$ and $Q : [X_0] \to [X_0]$ the bilipschitz map induced by q. Then $Q(F_\omega) = \cup_{i=0}^{p} W_i^\omega$. We denote $Int\, W_i^\omega$

the topological interior of W_i^ω in any apartment containing it. We denote $\mathfrak{O}' = \cup_{i=0}^p Int\, W_i^\omega$ and $\mathfrak{O} = Q^{-1}(\mathfrak{O}')$. The set $\mathfrak{O}$ is open in F_ω. The set $\mathfrak{O}'$ is locally isometric with an apartment. In particular for every $y' \in \mathfrak{O}'$ one may define the subspace of directions $(\mathfrak{O}')_{y'}$ in $\Sigma_{y'}\mathbf{K}$ and note that this is an apartment.

Let $\mathcal{S}_F^\omega = \cup_{m\in\mathbb{N}} \mathcal{S}_F^\omega(m)$, where $\mathcal{S}_F^\omega(m) = [\mathcal{S}_F(m)]$. Then

$$\forall m \in \mathbb{N}\,,\ \forall \rho > 0\,,\ \nu\big(\mathcal{S}_F^\omega \cap B^{F_\omega}(x_\omega, \rho)\big) \geq (1-\varepsilon_m)\nu(B^{F_\omega}(x_\omega,\rho))\,,$$

which implies that

$$\forall \rho > 0\,,\ \nu\big(\mathcal{S}_F^\omega \cap B^{F_\omega}(x_\omega,\rho)\big) = \nu\big(B^{F_\omega}(x_\omega,\rho)\big)\,.$$

Thus, almost every point of F_ω is in $\mathcal{S}_F^\omega$. Moreover, almost every point of F_ω is in $\mathcal{S}_F^\omega \cap \mathfrak{O}$.

Suppose $F_\omega = \prod_{j=1}^{\varkappa} \mathfrak{F}_j$, where $\mathfrak{F}_j$ is an apartment in $\mathbf{K}_j$. We denote $\mathfrak{F}_{\complement j} = \prod_{k=1,k\neq j}^{\varkappa} \mathfrak{F}_k$. We show that, up to a change of order between factors of the same rank in the target product $\mathbf{K} = \prod_{j=1}^{\varkappa} \mathbf{K}_j$, for every $j \in \{1,2,\dots,\varkappa\}$ there exists an apartment $\mathfrak{F}'_j$ of $\mathbf{K}_j$ such that for every $x_j \in \mathfrak{F}_{\complement j}$, $Q(\mathfrak{F}_j \times \{x_j\}) = \mathfrak{F}'_j \times \{x'_j\}$, where $x'_j \in \pi_j(Q(F_\omega))$. We write the full argument only for $j = 1$, for the other cases the arguments are analogous. Assume m is the maximal number in $\{1,2,\dots,\varkappa\}$ such that $r_1 = r_2 = \cdots = r_m$. The proof is done in several steps. First we show that near the points in $\mathfrak{O} \cap \mathcal{S}_F^\omega$, the map Q has special properties.

LEMMA 4.2.2. *Let y be a point in the set $\mathcal{S}_F^\omega \cap \mathfrak{O}$, $F_\omega = \cup_{j=0}^{p_0} \overline{W}_j^\omega$ the decomposition of F_ω into Weyl chambers of vertex y and $y' = Q(y) \in \mathfrak{O}'$. Then we have the following properties:*

(a) *The image of each Weyl chamber $Q(\overline{W}_j^\omega)$ coincides with one Weyl chamber of vertex y' near y';*

(b) *The image of the boundary of each Weyl chamber, $Q(\partial \overline{W}_j^\omega)$, near each point z' of it, coincides with a δ-Euclidean cone over a simplicial subcomplex of $\Sigma_{z'}\mathbf{K}$ homeomorphic to S^{r-2};*

(c) *Each $Q(\partial \overline{W}_j^\omega)$ is included into a finite union of singular hyperplanes, the cardinal of which is smaller than $N = N(L)$;*

(d) *The map Q induces a simplicial isomorphism between the apartments $(F_\omega)_y \subset \Sigma_y \mathbf{K}$ and $(\mathfrak{O}')_{y'} \subset \Sigma_{y'}\mathbf{K}$. In particular, in a neighborhood of y, Q sends singular (hyper)planes through y to singular (hyper)planes through y'.*

Proof. The point y is contained in $\mathcal{S}_F^\omega(m)$ for a certain $m \in \mathbb{N}$. So $y = [y_n]$ such that ω-almost surely $d(\bar{x}_0, \bar{y}_n) \leq mc_0$ and F is R_0m-l.b. with respect to y_n. This implies that each $\overline{W}_j^\omega$ occurs as the intersection of F_ω with another maximal flat F^j_ω contained in $[X_0]$. Then $Q(\overline{W}^\omega_j) = Q(F_\omega) \cap Q(F^j_\omega)$. Property (b) of the boundary of $\overline{W}^\omega_j$ then follows from Proposition 2.3.4 (ii), and topological considerations (as the boundary is locally a topologic disk).

Proposition 2.3.4 (i), implies that $Q(F_\omega) \subset \cup_{k=1}^s F_k$ and $Q(F^j_\omega) \subset \cup_{l=1}^t F'_l$, $s, t \leq M$. As proved previously, $Q(\partial\overline{W}^\omega_j)$ is, locally, a finite union of panels obtained as the intersection of two δ-Euclidean cones over bilipschitz spheres. The two δ-cones are contained in $\cup_{k=1}^s F_k$ and $\cup_{l=1}^t F'_l$, respectively. It follows that each panel has to be contained in a hyperplane which supports a Weyl polytope $F_k \cap F'_l$. Since there is only a finite number of such singular hyperplanes, property (c) follows.

In a neighborhood of y', $Q(F_\omega)$ coincides with $\mathfrak{D}'$ and $Q(F^j_\omega)$ with a δ-Euclidean cone of vertex y' over a bilipschitz sphere in $\Sigma_{y'}\mathbf{K}$. This implies that every $Q(\overline{W}^\omega_j)$ coincides, near y', with a finite union of Weyl chambers of vertex y' . This property, the fact that $Q(F_\omega)$ looks as an apartment near $y' \in \mathfrak{D}'$, and that $Q(\overline{W}^\omega_j) \cap Q(\overline{W}^\omega_k)$ has the same dimension as $\overline{W}^\omega_j \cap \overline{W}^\omega_k$, imply property (a). Moreover, in a neighborhood of y, Q sends singular (hyper)planes through y to singular (hyper)planes through y'. In particular Q induces a simplicial isomorphism between the apartments $(F_\omega)_y \subset \Sigma_y\mathbf{K}$ and $(\mathfrak{D}')_{y'} \subset \Sigma_{y'}\mathbf{K}$. □

Almost every point of F_ω is in $\mathcal{S}^\omega_F \cap \mathfrak{D}$. Then for a.e. $y_1 \in \mathfrak{F}_{\complement 1}$, a.e. $y \in \mathfrak{F}_1 \times \{y_1\}$ is in $\mathcal{S}^\omega_F \cap \mathfrak{D}$. We fix such an $y_1 \in \mathfrak{F}_{\complement 1}$ and show that the image by Q of $\mathfrak{F}_1 \times \{y_1\}$ has special properties.

Lemma 4.2.3. *Let y_1 be a point in the factor $\mathfrak{F}_{\complement 1}$ such that a.e. $y \in \mathfrak{F}_1 \times \{y_1\}$ is in $\mathcal{S}^\omega_F \cap \mathfrak{D}$. Then*

$$Q\big(\mathfrak{F}_1 \times \{y_1\}\big) \subset \bigcup_{k=1}^{n} \overline{\mathfrak{F}}_k\,, \tag{4.2}$$

where $n \leq N_1(L)$ and each $\overline{\mathfrak{F}}_k$ is an apartment in a copy of one of the factors of rank r_1.

Proof. We fix a point y_0 in the full measure set $(\mathfrak{F}_1 \times \{y_1\}) \cap \mathcal{S}^\omega_F \cap \mathfrak{D}$. The singular r_1-plane $\mathfrak{F}_1 \times \{y_1\}$ decomposes into r_1-dimensional walls of vertex y_0. Each of these walls appears as the intersection of the boundaries of two Weyl chambers in F_ω of vertex y_0, $\overline{W}^\omega_i$ and $\overline{W}^\omega_j$. By property (c) of Lemma 4.2.2, $Q(\partial\overline{W}^\omega_i) \subset \cup_{l=1}^s H_l$ and $Q(\partial\overline{W}^\omega_j) \subset \cup_{r=1}^t H'_r$, $s, t \leq N$. This and property (b) imply that the image by Q of the wall is included in a finite union of r_1-dimensional singular planes. These are the r_1-singular planes which support the Weyl polytopes $H_l \cap H'_r$. Thus

$$Q\big(\mathfrak{F}_1 \times \{y_1\}\big) \subset \bigcup_{j=1}^{v} \mathcal{H}_j \,, \tag{$*$}$$

where $\mathcal{H}_j$ is an r_1-dimensional singular plane and $v \leq N_1(L)$.

Almost every $y \in \mathfrak{F}_1 \times \{y_1\}$ has the property that in a neighborhood of $y' = Q(y)$, $Q\left(\mathfrak{F}_1 \times \{y_1\}\right)$ is entirely contained in one singular plane $\mathcal{H}_j$. To see this it suffices to notice that the set of points y not verifying the previous property is the preimage by Q of a set of dimension $r_1 - 1$. We denote the previous full measure set in $\mathfrak{F}_1 \times \{y_1\}$ by $\mathfrak{D}_1$.

In the sequel we fix a generic point $y \in \mathcal{S}_F^{\omega} \cap \mathfrak{D} \cap \mathfrak{D}_1$. On one hand, $Q(F_\omega)$ coincides with an apartment near y' and $Q : F_\omega \to Q(F_\omega)$ induces a simplicial isomorphism near y and y'. We know that each apartment in $\mathbf{K}$ decomposes into a product of apartments in the irreducible factors. In particular the map Q preserves the decompositions into products of apartments near y and y' or eventually inverts the order between apartments of the same dimension. Thus there exists $i \in \{1, 2, \ldots, m\}$ and $\mathfrak{F}_i(y)$ apartment of $\mathbf{K}_i$ such that for a neighborhood U_y of y, $Q\left(\left(\mathfrak{F}_1 \times \{y_1\}\right) \cap U_y\right) = \left(\mathfrak{F}_i(y) \times \{c_y\}\right) \cap V_{y'}$, where $V_{y'}$ is a neighborhood of y' and $c_y \in \pi_i(\mathfrak{D}')$.

On the other hand, since $y \in \mathfrak{D}_1$, by eventually taking U_y and $V_{y'}$ smaller, we have that $Q\left(\left(\mathfrak{F}_1 \times \{y_1\}\right) \cap U_y\right) = \mathcal{H}_j \cap V_{y'}$, for a certain j in $\{1, 2, \ldots, v\}$. This implies that $\mathcal{H}_j$ coincides, in a small ball, with an apartment in an irreducible r_1-rank factor. Then $\mathcal{H}_j$ is itself entirely contained in a r_1-rank factor, and since it is a r_1-singular plane, then it must be an apartment in this factor.

We select among the singular planes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_v$ the ones which are apartments in copies of r_1-rank factors. We denote them by $\overline{\overline{\mathfrak{F}}}_1, \overline{\overline{\mathfrak{F}}}_2, \ldots \overline{\overline{\mathfrak{F}}}_n$. By the previous argument we have obtained that for almost every $y \in \mathfrak{F}_1 \times \{y_1\}$, $Q(y)$ is contained in $\cup_{k=1}^{n} \overline{\overline{\mathfrak{F}}}_k$. Since a full measure subset in $\mathfrak{F}_1 \times \{y_1\}$ is dense in $\mathfrak{F}_1 \times \{y_1\}$ and the set $\cup_{k=1}^{n} \overline{\overline{\mathfrak{F}}}_k$ is closed, we obtain the inclusion (4.2). $\square$

Now we prove that for every $y_1 \in \mathfrak{F}_{\mathtt{C}1}$, $\pi_1(Q(\mathfrak{F}_1 \times \{y_1\}))$ is a point (up to inversion in the order of the factors of the same rank, in the target building).

LEMMA 4.2.4. *There exists $i \in \{1, 2, \ldots, m\}$ such that for every $y_1 \in \mathfrak{F}_{\mathtt{C}1}$, $\pi_i(Q(\mathfrak{F}_1 \times \{y_1\}))$ is a point.*

Proof. STEP 1. Let $y_1 \in \mathfrak{F}_{\mathtt{C}1}$ be so that a.e. $y \in \mathfrak{F}_1 \times \{y_1\}$ is in $\mathcal{S}_F^{\omega} \cap \mathfrak{D}$. We show that for every point y in $\mathfrak{F}_1 \times \{y_1\}$ there exist $i \in \{1, 2, \ldots, m\}$

and $c \in \pi_i(Q(\mathfrak{F}_1 \times \{y_1\}))$ such that

$$\exists U_y \text{ neighborhood of } y \text{ in } \mathfrak{F}_1 \times \{y_1\} \text{ satisfying } \pi_i(Q(U_y)) = c\,. \qquad (*)_{i,c}$$

In the proof of the previous lemma we have shown this only for a.e. y in $\mathfrak{F}_1 \times \{y_1\}$. Now we generalize the result to all points. Let y be a point in $\mathfrak{F}_1 \times \{y_1\}$ and $y' = Q(y)$. With an argument similar to the one in the proof of [KlL, Corollary 6.2.3] we deduce from (4.2) that there exists $\delta > 0$ such that $Q\left(\mathfrak{F}_1 \times \{y_1\}\right) \cap B(y', \delta) = \cup_{\ell=1}^{s} \mathcal{M}_\ell \cap B(y', \delta)$, where $\mathcal{M}_\ell$ is an r_1-dimensional wall of vertex y' and $\cup_{\ell=1}^{s} \mathcal{M}_\ell \cap B(y', \delta)$ is a δ-Euclidean cone over an $(r_1 - 1)$-dimensional sphere of $\Sigma_{y'}\mathbf{K}$.

On the other hand, by (4.2), the set $Q\left(\mathfrak{F}_1 \times \{y_1\}\right) \cap B(y', \delta)$ is contained in $\cup_{k=1}^{n} \overline{\mathfrak{F}}_k \cap B(y', \delta)$. In the space of directions $\Sigma_{y'}\mathbf{K}$ this inclusion becomes the inclusion of the $(r_1 - 1)$-dimensional sphere $\cup_{\ell=1}^{s} (\mathcal{M}_\ell)_{y'}$ into $\cup_{k=1}^{n} (\overline{\mathfrak{F}}_k)_{y'}$. The decomposition of $\mathbf{K}$ as a product of irreducible factors $\mathbf{K} = \prod_{j=1}^{\varkappa} \mathbf{K}_j$ implies that the space of directions $\Sigma_{y'}\mathbf{K}$ decomposes as a join of the spherical buildings $\Sigma_{y'_j}\mathbf{K}_j$, where $y' = (y'_1, y'_2, \dots y'_\varkappa)$, $y'_j \in \mathbf{K}_j$. Each $(\overline{\mathfrak{F}}_k)_{y'}$ is entirely contained in one of the spherical buildings $\Sigma_{y'_j}\mathbf{K}_j$ with $j \in \{1, 2, \dots m\}$. Hence the $(r_1 - 1)$-dimensional sphere of directions $\cup_{\ell=1}^{s} (\mathcal{M}_\ell)_{y'}$ is contained in the disjoint union of the spherical buildings $\Sigma_{y'_j}\mathbf{K}_j$ with j varying in $\{1, 2, \dots m\}$. But in this disjoint union each spherical building $\Sigma_{y'_j}\mathbf{K}_j$ is a connected component. It follows that $\cup_{\ell=1}^{s} (\mathcal{M}_\ell)_{y'}$ is entirely contained in only one building $\Sigma_{y'_j}\mathbf{K}_j$. Since $\cup_{\ell=1}^{s} \mathcal{M}_\ell \cap B(y', \delta)$ is a δ-Euclidean cone over the sphere of directions $\cup_{\ell=1}^{s} (\mathcal{M}_\ell)_{y'}$, it follows that it is contained in $\mathbf{K}_j \times \{\pi_j(y')\}$. The desired neighborhood of y is then $U_y = \mathfrak{F}_1 \times \{y_1\} \cap Q^{-1}(B(y', \delta))$.

Step 2. The next step is to show that, for the fixed y_1, every point y in $\mathfrak{F}_1 \times \{y_1\}$ satisfies the relation $(*)_{i,c}$ for the same i and the same c. We denote $\mathcal{O}_{i,c} = \{y \in \mathfrak{F}_1 \times \{y_1\} \mid y \text{ verifies } (*)_{i,c}\}$, for a certain $i \in \{1, 2, \dots, m\}$ and $c \in \pi_i(Q(\mathfrak{F}_1 \times \{y_1\}))$.

The family of sets $\mathcal{O}_{i,c}$, $i \in \{1, 2, \dots, m\}$, is a partition of open sets of $\mathfrak{F}_1 \times \{y_1\}$, which is a connected set. All the sets of the partition must then be empty, except one, which covers $\mathfrak{F}_1 \times \{y_1\}$. Thus there exists $i \in \{1, 2, \dots, m\}$ such that $\pi_i\left(Q\left(\mathfrak{F}_1 \times \{y_1\}\right)\right)$ is a point.

Step 3. We have obtained that for a.e. $y_1 \in \mathfrak{F}_{\complement 1}$, there exist $i \in \{1, 2, ..., m\}$ such that $\pi_i\left(Q\left(\mathfrak{F}_1 \times \{y_1\}\right)\right)$ is a point. If $y_n \to y_1$, $y_n, y_1 \in \mathfrak{F}_{\complement 1}$, then $Q\left(\mathfrak{F}_1 \times \{y_n\}\right)$ converges to $Q\left(\mathfrak{F}_1 \times \{y_1\}\right)$ in the modified Hausdorff metric [G1, § 6]. This implies that for every $y_1 \in \mathfrak{F}_{\complement 1}$, there exists $i \in \{1, 2, \dots, m\}$ and a neighborhood U of y_1 in $\mathfrak{F}_{\complement 1}$ such that $\forall y \in U$,

$\pi_i\left(Q\left(\mathfrak{F}_1 \times \{y\}\right)\right)$ is a point. Since $\mathfrak{F}_{\mathtt{C}1}$ is connected, there is an unique $i \in \{1, 2, \ldots, m\}$ such that for every $y_1 \in \mathfrak{F}_{\mathtt{C}1}$, $\pi_i\left(Q\left(\mathfrak{F}_1 \times \{y\}\right)\right)$ is a point. □

By eventually changing the order of the same rank factors in the target building we may suppose that the i in Lemma 4.2.4 is equal to 1.

Lemma 4.2.5. *There exists $\mathfrak{F}_1'$ apartment in $\mathbf{K}_1$ such that $\forall x_1 \in \mathfrak{F}_{\mathtt{C}1}$, $Q(\mathfrak{F}_1 \times \{x_1\}) = \mathfrak{F}_1' \times \{x_1'\}$, for some $x_1' \in \pi_1(Q(F_\omega))$.*

Proof. We first show that for every $x_1 \in \mathfrak{F}_{\mathtt{C}1}$, $Q\left(\mathfrak{F}_1 \times \{x_1\}\right) = \mathfrak{F}_1' \times \{x_1'\}$, for some $x_1' \in \pi_1(Q(F_\omega))$ and $\mathfrak{F}_1'$ apartment in $\mathbf{K}_1$ (in this step we admit that $\mathfrak{F}_1'$ might be different for each x_1).

It suffices if we prove the previous statement for a.e. $x_1 \in \mathfrak{F}_{\mathtt{C}1}$. For a.e. $x_1 \in \mathfrak{F}_{\mathtt{C}1}$ a.e. $x \in \mathfrak{F}_1 \times \{x_1\}$ is in $\mathcal{S}_F^\omega \cap \mathfrak{O}$. We fix such an x_1. By the previous Lemma $Q\left(\mathfrak{F}_1 \times \{x_1\}\right) \subset \mathbf{K}_1 \times \{x_1'\}$, $x_1' \in \pi_1(Q(F_\omega))$. In the sequel we shall "forget" the point factor. By (4.2),

$$Q\left(\mathfrak{F}_1\right) \subset \bigcup_{k=1}^{n} \overline{\mathfrak{F}}_k\ , \tag{4.3}$$

where $\overline{\mathfrak{F}}_k$ is an apartment in $\mathbf{K}_1$ and $n \leq N_1(L)$. Also, a.e. $x \in \mathfrak{F}_1$ verifies

(∗) $\exists U_x, U_{x'}$ neighborhoods of x and $x' = Q(x)$, and $\mathfrak{F}_1'(x)$ apartment of $\mathbf{K}_1$ such that $Q(\mathfrak{F}_1 \cap U_x) = U_{x'} \cap \mathfrak{F}_1'(x)$, and Q sends singular k-planes of $\mathfrak{F}_1$ through x to singular k-planes of $\mathfrak{F}_1'(x)$ through x'.

Due to (4.3) the set of apartments $\mathfrak{F}_1'(x)$ may be supposed finite.

A set homeomorphic and locally isometric to $\mathbb{R}^{r_1}$ is isometric to $\mathbb{R}^{r_1}$. Thus, to end the proof of this step, it suffices to show (∗) for every $x \in \mathfrak{F}_1$. By (4.3) and an argument similar to the one in the proof of [KlL, Corollary 6.2.3], for every $x \in \mathfrak{F}_1$ there exists $\delta > 0$ such that if $x' = Q(x)$, $Q(\mathfrak{F}_1) \cap B(x', \delta) = \cup_{\ell=1}^{s} \mathfrak{W}_\ell \cap B(x', \delta)$, where $\cup_{\ell=1}^{s} \mathfrak{W}_\ell \cap B(x', \delta)$ is a δ-Euclidean cone over a bilipschitz sphere of $\Sigma_{x'}\mathbf{K}_1$. Let s_0 be the number of chambers in an apartment of $\Sigma_{x'}\mathbf{K}_1$. We have $s \geq s_0$. To show that $\cup_{\ell=1}^{s} \mathfrak{W}_\ell \cap B(x', \delta)$ is the germ of an apartment it suffices , by Lemma 2.2.3 (ii), to prove $s = s_0$.

We choose and fix $\ell \in \{1, 2, \ldots, s\}$. Let $\mathfrak{O}_\ell' := Int\ \mathfrak{W}_\ell \cap B(x', \delta)$ and $\mathfrak{O}_\ell := Q^{-1}(\mathfrak{O}_\ell')$. The sets $\mathfrak{O}_\ell$ and $\mathfrak{O}_\ell'$ can both be seen as open subsets of $\mathbb{R}^{r_1}$. The map $Q_\ell := Q|_{\mathfrak{O}_\ell} : \mathfrak{O}_\ell \to \mathfrak{O}_\ell'$ is bilipschitz, so it is differentiable almost everywhere. Almost everywhere (∗) is verified. It follows that for a.e. $y \in \mathfrak{O}_\ell$, $d_y Q_\ell$ exists and it induces a simplicial isomorphism between $(\mathfrak{F}_1)_y$ and $(\mathfrak{O}_\ell')_{y'}$, where $y' = Q(y)$. In particular, $d_y Q_\ell$ sends singular hyperplanes to singular hyperplanes. Since $\mathbf{K}_1$ is an irreducible Euclidean building of rank $r_1 \geq 2$, there are at least $r_1 + 1$ singular hyperplanes in the associated Coxeter complex. It follows that d_yQ_ℓ is a similarity (by similarity we mean isometry up to rescaling). More precisely d_yQ_ℓ is a homothety eventually composed with an element of the Coxeter group. We conclude that for a.e. $y \in \mathfrak{O}_\ell$, d_yQ_ℓ exists and is a similarity. Thus $Q_\ell : \mathfrak{O}_\ell \to \mathfrak{O}'_\ell$ is bilipschitz and 1-quasi-conformal [P, §7.10]. Hence Q_ℓ is a C^∞-map and $\forall y \in \mathfrak{O}_\ell$, d_yQ_ℓ is a similarity. Since for a.e. y, d_yQ_ℓ sends singular hyperplanes to singular hyperplanes and the differential dQ_ℓ is continuous, we may conclude that for every y, d_yQ_ℓ sends singular hyperplanes to singular hyperplanes. Then, two differentials of Q_ℓ, d_yQ_ℓ and d_zQ_ℓ, in two different points $y, z \in \mathfrak{O}_\ell$, differ by a composition to the left with an element of Cox, the Coxeter group associated to the Euclidean building $\mathbf{K}_1$, and with a homothety. The continuity of dQ_ℓ and the finiteness of Cox implies that all similarities d_yQ_ℓ differ only by compositions to the left with homotheties. In particular Q_ℓ preserves all foliations by parallel singular hyperplanes or lines, so Q_ℓ is a homothety [KlL, §6.4.4]. The sets $\mathfrak{O}_\ell$ and $\mathfrak{O}'_\ell$ are open and dense in $Q^{-1}(\mathfrak{W}_\ell \cap B(x', \delta))$ and $\mathfrak{W}_\ell \cap B(x', \delta)$, respectively. It follows that $Q : Q^{-1}(\mathfrak{W}_\ell \cap B(x', \delta)) \to \mathfrak{W}_\ell \cap B(x', \delta)$ is an homothety. Since the previous reasoning can be done for every $\ell \in \{1, 2, \dots, s\}$, we may conclude that $Q : \mathfrak{F}_1 \cap Q^{-1}(B(x', \delta)) \to \cup_{\ell=1}^{s} \mathfrak{W}_\ell \cap B(x', \delta)$ is an homothety, where $\cup_{\ell=1}^{s} \mathfrak{W}_\ell \cap B(x', \delta)$ is endowed with the length metric. We conclude that $s = s_0$ and $\cup_{\ell=1}^{s} \mathfrak{W}_\ell \cap B(x', \delta)$ is the germ of an apartment.

We now show the independence of the apartment $\mathfrak{F}'_1$ obtained in the previous step, of the point $x_1 \in \mathfrak{F}_{\mathtt{C1}}$. We denote the set of apartments $\mathfrak{F}'_1$ of $\mathbf{K}_1$ corresponding to different points $x_1 \in \mathfrak{F}_{\mathtt{C1}}$ by Ap_1. If $x_n, x_1 \in \mathfrak{F}_{\mathtt{C1}}$, $x_n \to x_1$, then $Q(\mathfrak{F}_1 \times \{x_n\})$ converges to $Q(\mathfrak{F}_1 \times \{x_1\})$ in the modified Hausdorff metric. Thus we may conclude that for every $x_1 \in \mathfrak{F}_{\mathtt{C1}}$ there exists $\mathfrak{F}'_1 \in Ap_1$ and an open neighborhood V_{x_1} of x_1 in $\mathfrak{F}_{\mathtt{C1}}$ such that $\forall y \in V_{x_1}$, $Q(\mathfrak{F}_1 \times \{y\}) = \mathfrak{F}'_1 \times \{y'\}$, where $y' \in \pi_1(Q(F_\omega))$. The class of sets $\mathcal{O}(\mathfrak{F}'_1) = \{x_1 \in \mathfrak{F}_{\mathtt{C1}} \mid \exists V_{x_1}$ neighborhood of x_1 such that $\forall y \in V_{x_1}$, $Q(\mathfrak{F}_1 \times \{y\}) = \mathfrak{F}'_1 \times \{y'\}\}$, $\mathfrak{F}'_1 \in Ap_1$, is an open partition of $\mathfrak{F}_{\mathtt{C1}}$, or $\mathfrak{F}_{\mathtt{C1}}$ is connected. Then it exists $\mathfrak{F}'_1 \in Ap_1$ such that $\forall x_1 \in \mathfrak{F}_{\mathtt{C1}}$, $Q(\mathfrak{F}_1 \times \{x_1\}) = \mathfrak{F}'_1 \times \{x'_1\}$, for some x'_1. □

With analogous arguments we obtain the same result as in Lemma 4.2.5 for every $i \in \{2, 3, \dots, \varkappa\}$.

End of the proof of Proposition 4.2.1. It is easy to deduce from the previous results that $Q(F_\omega)$ is an apartment. One can prove by induction on $k \in \{1, 2, \dots, \varkappa\}$ that $\forall x_k \in \prod_{i=k+1}^{\varkappa} \mathfrak{F}_i$ there exists an $x'_k \in \prod_{i=k+1}^{\varkappa} \mathbf{K}_i$ such that $Q\big(\prod_{i=1}^{k} \mathfrak{F}_i \times \{x_k\}\big) = \prod_{i=1}^{k} \mathfrak{F}'_i \times \{x'_k\}$. Finally we obtain that $Q(F_\omega) =$

$\cup_{i=0}^{p} W_i^\omega$ is an apartment, so $p = p_0$ and $\cup_{i=0}^{p} W_i(\infty)$ is an apartment. As F is an l.b. flat with respect to x, if $F = \cup_{j=0}^{p_0} \widetilde{W}_j$ is the decomposition of F into Weyl chambers of vertex x, each $\widetilde{W}_j$ is l.i. with respect to x. Corollary 3.3.10 implies that there exists a partition $\sqcup_{j=0}^{p_0} I_j$ of $\{0, 1, \dots, p_0\}$ such that for each $j \in \{0, 1, \dots, p_0\}$, $q(\pi_0(\widetilde{W}_j))$ has the same horizon as $\cup_{i \in I_j} W_i$. Each set I_j must contain only one element which we denote i_j. In any asymptotic cone we have that $Q(\widetilde{W}_j^\omega) = W_{i_j}^\omega$. Then, if $\widetilde{W}_j^\omega$ and $\widetilde{W}_k^\omega$ have a wall in common, $W_{i_j}^\omega$ and $W_{i_k}^\omega$ have in common a wall of the same dimension. Then the same is true in the space: if $\widetilde{W}_j$ and $\widetilde{W}_k$ have a wall $\mathcal{M}$ in common, W_{i_j} and W_{i_k} have in common a wall $\mathcal{M}'$, of the same dimension. The simplicial homomorphism between the apartments $F(\infty)$ and $\cup_{i=0}^{p} W_i(\infty)$ associating to the boundary at infinity of each such wall $\mathcal{M}(\infty)$ the boundary at infinity $\mathcal{M}'(\infty)$ preserves all incidence relations so it is an isomorphism. We have thus proved the last part of the statement. □

We have an uniformity result that completes Lemma 3.3.7 under stronger assumptions. We prove that for an R-logarithmic flat to which a quasi-isometry associates a fan over an apartment the distance R_ε starting from which its image and the fan are seen ε-close depends only on R and the constants of the quasi-isometry. In Lemma 3.3.7 only the closeness of the image of the flat to the fan was taken into account, but not the one of the fan to the image of the flat.

Proposition 4.2.6 (uniformity). *Let $\varepsilon > 0, R > 0, L \geq 1$ and $c > 0$. There exists $R_\varepsilon = R_\varepsilon(R, L, c)$ such that if F is an R–l. flat with respect to x and $q : X_0 \to X_0$ is an (L, c)-quasi-isometry, $q(x) = x$, and if q associates to F a fan over an apartment of vertex x, $\cup_{i=0}^{p_0} W_i$, then for every $\rho \geq R_\varepsilon$ and $\xi = \varepsilon/100$ we have*

$$\tilde{\angle}_x^H \big(q(\pi_0(F)) \cap \mathfrak{C}_\xi(x, \rho), \bigcup_{i=0}^{p_0} W_i \cap \mathfrak{C}_\xi(x, \rho) \big) \leq \varepsilon \,.$$

Proof. We argue by contradiction and suppose that for certain $\varepsilon > 0$, $R > 0$, $L \geq 1$ and $c > 0$ there exists a sequence (F_n) of R-l. flats with respect to x and a sequence of (L, c)-quasi-isometries $q_n : X_0 \to X_0$, $q_n(x) = x$, such that if $\cup_{i=0}^{p_0} W_i^n$ are fans over apartments associated to F_n by q_n and if

$$R_n^\varepsilon := \sup \big\{ \rho > 0 \bigm| \tilde{\angle}_x^H (q_n(\pi_0(F_n)) \cap \mathfrak{C}_\xi(x, \rho), \bigcup_{i=0}^{p_0} W_i^n \cap \mathfrak{C}_\xi(x, \rho)) > \varepsilon \big\} \,,$$

then $R_n^\varepsilon \to \infty$. In $\mathbf{K} = Con_\omega(X, x, (R_n^\varepsilon))$ we have that

$$Q(F_\omega) = \big[q_n(\pi_0(F_n))\big] \subset \left[\bigcup_{i=0}^{p_0} W_i^n\right] = \bigcup_{i=0}^{p_0} W_i^\omega \tag{4.4}$$

by Corollary 3.3.8, and there exists an element $z_\omega \in \cup_{i=0}^{p_0} W_i^\omega \cap \mathfrak{C}_\xi(x_\omega, 1)$ such that

$$\tilde{\angle}_{x_\omega}\big(z_\omega, Q(F_\omega) \cap \mathfrak{C}_\xi(x_\omega, 1)\big) \geq \varepsilon\,.$$

For a small δ, $Q(F_\omega) \cap B(x_\omega, \delta)$ is a δ-Euclidean cone over a bilipschitz sphere of $\Sigma_{x_\omega}\mathbf{K}$ and it is included in $\cup_{i=0}^{p_0} W_i^\omega \cap B(x_\omega, \delta)$. By Lemma 2.2.3 (i) and (ii), $Q(F_\omega) \cap B(x_\omega, \delta)$ coincides with $\cup_{i=0}^{p_0} W_i^\omega \cap B(x_\omega, \delta)$ and it is a δ-Euclidean cone over a spherical apartment. It follows that $\cup_{i=0}^{p_0} W_i^\omega$ is an Euclidean cone over a spherical apartment, so a maximal flat. Then in (4.4) we have equality. This contradicts the existence of z_ω. □

Proposition 4.2.7. *Let $q : X_0 \to X_0$ be an (L, c)-quasi-isometry, $q(x) = x$, and F an R-l. flat with respect to x to which q associates a fan over an apartment of vertex x, $\cup_{i=0}^{p_0} W_i$. If F' is the maximal flat asymptotic to the fan, then $d(x, F') \leq C$, where C is a constant depending only on R, c and L.*

Proof. Suppose, on the contrary, that there exists a sequence F_n of R-l. flats with respect to x and a sequence $q_n : X_0 \to X_0$ of (L, c)-quasi-isometries, $q_n(x) = x$, such that $\delta_n := d(x, F_n') \to +\infty$. In $Con_\omega(X, x, \delta_n)$ we may deduce as in the proof of Proposition 4.2.6 that $[\cup_{i=0}^{p_0} W_i^n]$ is a maximal flat, which we denote by F_ω, and that, since the bilipschitz flat $[q_n(\pi_0(F_n))]$ is contained in it, it coincides with it.

On the other hand, since the Hausdorff distance between $\cup_{i=0}^{p_0} W_i^n$ and F_n' is equal to $\delta_n = d(x, F_n')$, in the cone the maximal flats F_ω and $F_\omega' = [F_n']$ are at Hausdorff distance 1. This implies that $F_\omega = F_\omega'$ by [KlL, Corollary 4.6.4]. But since $d(x, F_n') = \delta_n$, the limit point x_ω, which is contained in F_ω, is at distance 1 from F_ω'. We have obtained a contradiction. □

Remark 4.2.8. (1) *We may replace everywhere the hypothesis $q(x) = x$ by $q(x) = y$, where y is an arbitrary point in X_0. The fan $\cup_{i=0}^{p_0} W_i$ is then a fan of vertex y. In Proposition 4.2.7 we obtain $d(y, F') \leq C$.*

(2) *Let F be a good l.b. flat with respect to $x \in F \cap X_0$. Let $q : X_0 \to X_0$ be an (L, c)-quasi-isometry, $\cup_{i=0}^{p_0} W_i$ be the fan over an apartment of vertex $q(x)$ associated to F by q and F' be the maximal flat asymptotic to $\cup_{i=0}^{p_0} W_i$. Each subset $\mathcal{S}_F(m)$, which is a big subset of F if c_0, R_0 and $m \in \mathbb{N}$ are big, is sent by $q \circ \pi_0$ at finite distance from F': $q \circ \pi_0(\mathcal{S}_F(m)) \subset N_{C_m}(F')$.*

Proof. The proof of (1) is done with the same arguments as in the Remark 3.3.11. The statement (2) is a consequence of (1) and Proposition 4.2.7. □

The result (2) in the previous remark is similar to Theorem 8.1 obtained by A. Eskin and B. Farb [EF], though less general.

Before stating the following result we recall some simple properties. Let Λ be a subset of s roots in the chosen fundamental set of roots Δ_0. If $k \in K_\Lambda$, then the maximal flats F_0 and kF_0 have at least the codimension s singular plane $H_\Lambda = \cap_{\alpha\in\Lambda} \ker\alpha$ in common. If moreover $k \notin \cup_{\Lambda_1\subset\Lambda} K_{\Lambda_1}$ then H_Λ is precisely the intersection between F_0 and kF_0.

LEMMA 4.2.9. *Let $q : X_0 \to X_0$ be an (L, c)-quasi-isometry, $q(x) = x$. Let Λ be a subset of cardinal s in Δ_0. Suppose that $g \in G$ and $k \in K_\Lambda \setminus \cup_{\Lambda_1\subset\Lambda} K_{\Lambda_1}$ have the property that the maximal flats $F^0 = gF_0$ and $F^1 = gkF_0$ are R-logarithmic with respect to $x = gx_0$, and q associates to each of them a fan over an apartment. Let $\widetilde{F}^0$ and $\widetilde{F}^1$ be the maximal flats asymptotic to those fans. Then*

(i) *The boundaries at infinity $\widetilde{F}^0(\infty)$ and $\widetilde{F}^1(\infty)$ intersect in a singular plane of codimension s, $\mathcal{H}$.*

(ii) *Let x_j be the projection of x on $\widetilde{F}^j$ and H_j be the singular plane through x_j with boundary at infinity $\mathcal{H}$, $j = 0, 1$. Then there exists $C = C(R, L, c)$ such that*

$$x \in N_C(H_0) \cap N_C(H_1) \text{ and } H_i \subset N_{2C}(H_j)\,,\ \{i, j\} = \{0, 1\}\,.$$

(iii) *For every $\delta > 0$ there exists $D = D(R, L, c, \delta)$ such that if $d_{\overline{K}_\Lambda}(\bar{k}, \cup_{\Lambda_1\subset\Lambda}\overline{K}_{\Lambda_1}) \geq \delta$, then*

$$q\big(\mathcal{S}_{F^0}(1) \cap \mathcal{S}_{F^1}(1)\big) \subset N_D(H_0) \cap N_D(H_1)\,.$$

Proof. (i) Let $H := gH_\Lambda$ be the singular codimension s plane in which F^0 and F^1 intersect. In an asymptotic cone, let H^ω be the limit set of H. By Proposition 4.2.7 we have that the limit flat $(\widetilde{F}^i)^\omega$ coincides with $Q((F^i)^\omega)$ for $i = 0, 1$. Then $(\widetilde{F}^0)^\omega \cap (\widetilde{F}^1)^\omega$ coincides with $Q(H^\omega)$. The previous set is bilipschitz equivalent to $\mathbb{R}^{r-s}$ and it is a Weyl polytope, as intersection of two apartments [KlL, Corollary 4.4.6]. The only Weyl polytopes which are bilipschitz equivalent to $\mathbb{R}^{r-s}$ are codimension s singular planes. So $(\tilde{F}^0)^\omega \cap (\tilde{F}^1)^\omega$ is a codimension s singular plane. Property (i) then follows from Lemma 2.4.3.

(ii) The fact that $d(x, x_j) \leq C,\ j = 0, 1$, follows from Proposition 4.2.7. Then $d(x_0, x_1) \leq 2C,\ x_j \in H_j,\ j = 0, 1$, and H_0, H_1 are asymptotic. This implies the second part of (ii).

(iii) Suppose, on the contrary, that there exist $\delta > 0$, two sequences of maximal flats $F_n^0 = g_n F_0$ and $F_n^1 = g_n k_n F_0$ with $d_{\overline{K}_\Lambda}\left(\bar{k}_n, \cup_{\Lambda_1 \subset \Lambda} \overline{K}_{\Lambda_1}\right) \geq \delta$, and a sequence of (L,c)-quasi-isometries q_n satisfying the hypothesis, and there exist $z_n \in q_n\left(\mathcal{S}_{F_n^0}(1) \cap \mathcal{S}_{F_n^1}(1)\right)$ with $\max\{d(z_n, H_{0n}), d(z_n, H_{1n})\} = D_n \to \infty$. Let $H_n = F_n^0 \cap F_n^1$. In $Con_\omega(X, (z_n), (D_n))$ we obtain two apartments $F_\omega^j = [F_n^j]$ in $[X_0]$, $j = 0, 1$, which intersect in a codimension s singular plane $H_\omega = [H_n]$. Let Q be the bilipschitz map induced by the sequence of quasi-isometries (q_n). The image by Q of the singular plane, $Q(H_\omega) = Q(F_\omega^0) \cap Q(F_\omega^1)$ coincides with $(\widetilde{F}^0)^\omega \cap (\widetilde{F}^1)^\omega$ by Propositions 4.2.6 and 4.2.7. The set $Q(H_\omega) = (\widetilde{F}^0)^\omega \cap (\widetilde{F}^1)^\omega$ contains, by (ii), the singular plane $H'_\omega = [H_{jn}]$, $j = 0, 1$, and also a point $z_\omega = [z_n]$ at distance 1 from H'_ω. Since the set $Q(H_\omega)$ is a bilipschitz flat of dimension $r - s$ and it contains the flat of dimension $r - s$ H'_ω, the two sets should coincide. This contradicts the existence of the point z_ω. □

In the previous lemma too we may replace the hypothesis $q(x) = x$ by $q(x) = y$, $y \in X_0$, and we obtain $y \in N_C(H_0) \cap N_C(H_1)$.

5 The Associated Isometry. Conclusions

We fix a quasi-isometry q and we shall find an isometry which is within finite distance from it. We use Tits' theorem (Theorem 2.3.2).

In this section we identify the set of chambers in $\partial_\infty X$, which we denote by $Ch(\partial_\infty X)$, endowed with the cone topology, with the set of Weyl chambers of vertex x_0 with the modified Hausdorff topology, and with $\overline{K}$ with the induced topology.

By Proposition 4.2.1, q associates to each Weyl chamber in a full measure set of Weyl chambers of vertex x_0 one Weyl chamber. We denote this injective map between Weyl chambers $\tilde{q}$. Using $\tilde{q}$ and results of J. Tits we construct a simplicial isomorphism Φ on $\partial_\infty X$. We prove that Φ coincides with $\tilde{q}$ on a full measure set of chambers. To apply Tits' theorem, we must also prove that Φ is a homeomorphism. Since it is defined on a compact set, it suffices to prove that Φ is continuous. This shall be done by using the continuity of Φ on the chamber stars of all the panels contained in a given apartment. Finally, we show that the isometry g associated to Φ by Tits' theorem is at a finite distance from q.

5.1 Full measure subsets.

Lemma 5.1.1. *Let G be a group, H a subgroup and $\mathcal{E}$ a full measure subset in G. Then almost every g in G has the properties that $g \in \mathcal{E}$ and* *for almost every $h \in H$, $gh \in \mathcal{E}$.*

Proof. By Fubini theorem, almost every $\bar{g} \in G/H$ has the property that for almost every $h \in H$, $gh \in \mathcal{E}$. This implies the conclusion of the lemma. □

Let $\mathcal{L} := \{g \in G \mid gF_0$ is a good l.b. flat with respect to $gx_0\}$, which is a full measure subset of G. Let Λ be a subset of roots in Ξ. By Lemma 5.1.1, for a.e. $g \in G$, $g \in \mathcal{L}$ and $gk \in \mathcal{L}$ for a.e. $k \in K_\Lambda$. We denote $\mathcal{L}_\Lambda$ this new full measure subset and $\mathcal{B} := \cap_{\Lambda \subset \Xi} \mathcal{L}_\Lambda$.

The meaning of g being in $\mathcal{L}_\Lambda$ is that gF_0 is a good l.b. flat with respect to gx_0 and the same is true for almost every other maximal flat through gx_0 having in common with gF_0 the singular (hyper)plane gH_0^Λ. If g is in $\mathcal{B}$, then the previous statement is true for all singular (hyper)planes $gH_0^\Lambda, \Lambda \subset \Xi$, that is, for all singular (hyper)planes in gF_0 through gx_0.

Definition 5.1.2. *We call butterfly flat with respect to x a good l.b. flat with respect to x containing x, such that if H is any of its singular (hyper)planes through x, a.e. flat through H is a good l.b. flat with respect to x. The boundary at infinity of a butterfly flat is called a butterfly apartment.*

We notice that $g \in \mathcal{B}$ is equivalent to the fact that gF_0 is a butterfly flat. Since $\mathcal{B}$ is full measure in G, we conclude that for a.e. g, gF_0 is a butterfly flat. By Fubini theorem, this implies that for a.e. $x \in X$, a.e. maximal flat through x is a butterfly flat.

Let U_Ψ be an unipotent subgroup, $\Psi \subset \Xi$. For almost every $g \in G$, $g \in \mathcal{B}$ and $gu \in \mathcal{B}$ for a.e. $u \in U_\Psi$. We denote this new full measure set $\mathcal{B}_\Psi$ and $\mathcal{P} := \cap_{\Psi \subset \Xi} \mathcal{B}_\Psi$.

We recall that U_Ψ acts simply trasitively on the set of maximal flats whose boundaries at infinity contain $D_\Psi^+(\infty)$ (see subsection 2.5.A for definition of D_Ψ^+). Thus, if g is in $\mathcal{B}_\Psi$, this means that gF_0 is a butterfly flat with respect to gx_0 and a.e. maximal flat whose boundary at infinity contains $gD_\Psi^+(\infty)$ is a butterfly flat with respect to one of its points. If $g \in \mathcal{P}$, the previous property is true for all subsets of roots $\Psi \subset \Xi$.

Definition 5.1.3. *We call pistil flat with respect to gx_0 a butterfly flat $F = gF_0$ with respect to gx_0 with the property that for every $\Psi \subset \Xi$, a.e. maximal flat whose boundary at infinity contains $gD_\Psi^+(\infty)$ is a butterfly flat with respect to one of its points. The boundary at infinity of a pistil flat is called a pistil apartment.*

If $g \in \mathcal{P}$ then gF_0 is a pistil flat. The property of $\mathcal{P}$ being full measure implies that for almost every $x \in X$, almost every maximal flat through x

is a pistil flat.

We suppose we have chosen our fixed basepoint x_0 in the previous full measure subset of X. Thus the set $\{k \in K \mid kF_0$ is a pistil flat with respect to $x_0\}$ has full measure.

Let $F = gF_0$ be a pistil flat and $\mathcal{F} = F(\infty)$. Let $\mathcal{W}_1 = g\mathcal{W}_0$ and let $\mathcal{W}_2$ be another chamber in $\mathcal{F}$ and $gD^+_\Psi(\infty)$ the convex hull of $\mathcal{W}_1$ and $\mathcal{W}_2$. We recall that we denote by $op_{\mathcal{F}}$ the opposition isomorphism in $\mathcal{F}$ and by ϖ the composition $op_{\mathcal{F}} \circ \mathrm{retr}_{\mathcal{F},\mathcal{W}_1}$. By the discussion in subsection 2.5.A, there exists a bijection of the set of chambers $Op_{\mathcal{W}_1}(\mathcal{W}_2)$ onto U_Ψ. We consider $Op_{\mathcal{W}_1}(\mathcal{W}_2)$ endowed with the measure induced by this bijection.

Lemma 5.1.4. *For almost every chamber $\mathcal{W}$ in $Op_{\mathcal{W}_1}(\mathcal{W}_2)$, every chamber $\mathcal{W}'$ in a minimal gallery from $\mathcal{W}_1$ to $\mathcal{W}$ determines with $\varpi(\mathcal{W}')$ a butterfly apartment.*

Proof. It is a consequence of the definition of a pistil flat and of the diffeomorphism defined in the end of subsection 2.5.A. □

Definition 5.1.5. *Let $F = gF_0$ be a pistil flat and $\mathcal{F} = F(\infty)$. Let $\mathcal{W}_1 = g\mathcal{W}_0$.*

With the previous notations, a chamber $\mathcal{W}$ such that every chamber $\mathcal{W}'$ in a minimal gallery from $\mathcal{W}$ to $\mathcal{W}_1$ determines with $\varpi(\mathcal{W}')$ a butterfly apartment, is called a chamber tied to the pistil apartment $\mathcal{F}$ with respect to the chamber $\mathcal{W}_1$.

We notice that if a chamber $\mathcal{W}$ is tied to a pistil apartment $\mathcal{F}$ with respect to a chamber $\mathcal{W}_1$, then every chamber in a minimal gallery from $\mathcal{W}$ to $\mathcal{W}_1$ is tied to $\mathcal{F}$ with respect to $\mathcal{W}_1$.

Corollary 5.1.6. *Let $\mathcal{F}$ be a pistil apartment and $\mathcal{W}_1, \mathcal{W}_2$ two chambers in it. Almost every chamber in $Op_{\mathcal{W}_1}(\mathcal{W}_2)$ is tied to $\mathcal{F}$ with respect to $\mathcal{W}_1$.*

5.2 The butterfly argument. In the sequel we suppose $q(x_0) = x_0$ and, unless otherwise stated, we always consider [good] l.[b.] flats and l.i. Weyl chambers, butterfly and pistil flats with respect to x_0.

Let $\mathbf{L}_K := \{k \in K \mid kF_0$ is a good l.b. flat $\}$ and $\mathbf{L}$ the projection of $\mathbf{L}_K$ in $\overline{K}$. We also denote by $\mathbf{L}$ the set of Weyl chambers $\{kW_0 \mid \bar{k} \in \mathbf{L}\}$.

By Proposition 4.2.1, we can define a map $\tilde{q}$ from $\mathbf{L}$ to $\overline{K}$ by $\tilde{q}(\bar{k}) = \bar{k}'$ so that $q(\pi_0(kW_0))$ has the same horizon as $k'W_0$ with respect to x_0. Using q^{-1} it is not difficult to see that $\tilde{q}$ is injective. Also, if $\bar{k}_1, \bar{k}_2 \in \mathbf{L}$ and $\bar{k}_1W_0 \cap \bar{k}_2W_0$ is a codimension m wall, by eventually using an asymptotic cone argument one can deduce that $\tilde{q}(\bar{k}_1)W_0 \cap \tilde{q}(\bar{k}_2)W_0$ is a codimension m wall. Thus, $\tilde{q}$ preserves all kinds of intersections.

Let $\mathcal{M}$ be a wall of vertex x_0. Suppose $\mathcal{M} = k\mathcal{M}_0$, where $k \in K$ and $\mathcal{M}_0 = W_0 \cap H_0^\Lambda$, for a certain $\Lambda \subset \Delta_0$. We may identify the chamber star of $\mathcal{M}$, $St\,\mathcal{M}$, with $k\overline{K}_\Lambda$.

Proposition 5.2.1. *Let $\mathcal{M}$ be a wall of vertex x_0. If $\mathbf{L} \cap St\,\mathcal{M}$ has full measure in $St\,\mathcal{M}$, then $\tilde{q}$ is uniformly continuous on $\mathbf{L} \cap St\,\mathcal{M}$.*

Proof. Step 1. In this step we choose and fix a pair of flats whose intersection is the minimal singular plane containing $\mathcal{M}$ and which satisfy the assumptions in Lemma 4.2.9. This choice and Lemma 4.2.9 will provide us with a singular plane of reference H'.

Let $H = kA_\Lambda x_0$ be the i-dimensional singular plane containing $\mathcal{M}$, where $k \in K$ and $\Lambda \subset \Delta_0$. We identify $St\,\mathcal{M}$ with $k\overline{K}_\Lambda$.

We may write $\mathbf{L} \cap St\,\mathcal{M} = k\overline{\mathcal{E}}$, where $\overline{\mathcal{E}}$ is a full measure subset of $\overline{K}_\Lambda$. Let $\mathcal{E}$ be the preimage of $\overline{\mathcal{E}}$, full measure subset of K_Λ. We choose an element k_0 in $\mathcal{E}$. For some fixed $\delta_0 > 0$, there exists $k_\Lambda \in \mathcal{E}$ such that $d_{\overline{K}_\Lambda}(\bar{k}_\Lambda, k_0(\cup_{\Lambda_1 \subset \Lambda}\overline{K}_{\Lambda_1})) > \delta_0$. Since $k\bar{k}_0$ and $k\bar{k}_\Lambda$ are in $\mathbf{L} \cap St\,\mathcal{M}$, the flats $F = kk_0F_0$ and $F_\Lambda = kk_\Lambda F_0$ are good R-l.b. flats for a certain $R \in \mathbb{R}_+^*$. The flats F and F_Λ intersect only in H, due to the way we have chosen k_Λ.

The quasi-isometry q associates to F and F_Λ fans over apartments. Let F' and F'_Λ be the maximal flats asymptotic to these fans. By Lemma 4.2.9 there exist $C = C(R, L, c)$ and $D = D(R, L, c)$ and a singular plane of dimension i, $H' \subset F'$, such that

$$x_0 \in N_C(H') \text{ and } q\big(\mathcal{S}_F(1) \cap \mathcal{S}_{F_\Lambda}(1)\big) \subset N_D(H'). \tag{5.1}$$

Step 2. Now we prove the statement of the proposition. Let W_1 and W_2 be two Weyl chambers in $\mathbf{L} \cap St\,\mathcal{M}$. There exist two good l.b. flats F_1 and F_2 containing W_1 and W_2. The flats F_1 and F_2 have at least H in common.

We fix an arbitrary $\varepsilon > 0$. We prove that there exists $\delta > 0$ small enough so that if W_1' and W_2' are the Weyl chambers of vertex x_0 associated to W_1 and W_2 by $\widetilde{q}$, $d_{\overline{K}}(W_1, W_2) < \delta$ implies $d_{\overline{K}}(W_1', W_2') \leq \varepsilon$. We recall that $d_{\overline{K}}$ is the K-invariant metric on $\overline{K}$. The outline of the proof is as follows. First we replace the distance $d_{\overline{K}}$ by the Hausdorff distance between traces of big balls. Then we notice that we may replace the basepoint x_0 by any other point y in the singular plane H. Propositions 4.2.1 and 4.2.6 allow to transfer any hypothesis of closeness between two Weyl chambers to the Weyl chambers associated to them by q. By choosing well the point y we may make sure that its image by q is not too far from the singular plane H' provided by Step 1. So up to a small displacement, we may suppose that the associated Weyl chambers have vertices on H'. Then we may slip again along H' and suppose that the two associated Weyl chambers have the vertex in the nearest point to x_0 of H'. By the relation (5.1), x_0 is not too far from H', so up to another bounded displacement, we obtain that the associated Weyl chambers with vertices in x_0 are close. This gives the conclusion.

For every $\delta_0 > 0$ small and R large, there exists a small δ such that

$$d_{\overline{K}}(W_1, W_2) < \delta \Longrightarrow d_{\mathcal{H}d}\big(W_1 \cap B(x_0, R), W_2 \cap B(x_0, R)\big) < \delta_0 R\,,$$

where by $d_{\mathcal{H}d}$ we denote the Hausdorff distance.

We take R big enough such that c_0 and R_0 chosen in the beginning of section 4.1, $LR_{\varepsilon/4}$, C, D and the constant c of the quasi-isometry q are very small compared to it.

Notation. Let W be a Weyl chamber and x a point. We denote by $W(x)$ the Weyl chamber of vertex x asymptotic to W.

We have

$$\begin{aligned} d_{\mathcal{H}d}&\big(W_1 \cap B(x_0, R), W_2 \cap B(x_0, R)\big) \\ &= d_{\mathcal{H}d}\big(W_1(y) \cap B(y, R), W_2(y) \cap B(y, R)\big) < \delta_0 R \end{aligned}$$

for every point $y \in H$. Since F_1 and F_2, as well as F and F_Λ chosen in Step 1, satisfy $(P_{x_0}(1))$, we can find a point $y \in H$ such that $y \in X_0$ and F_1, F_2, F and F_Λ are R_0-l.b. with respect to y. Since the Weyl chambers W_1 and W_2 are R_0-l.i. we may write

$$\begin{gathered} d_{\mathcal{H}d}\big(W_i(y) \cap \mathfrak{C}(y, R_0, \rho), \pi_0(W_i(y)) \cap \mathfrak{C}(y, R_0, \rho)\big) \leq M_0 \log \rho\,, \\ \forall \rho \geq R_0\,,\ i = 1, 2\,, \end{gathered}$$

and we may conclude that

$$d_{\mathcal{H}d}\big(\pi_0(W_1(y)) \cap \mathfrak{C}(y, R_0, R), \pi_0(W_2(y)) \cap \mathfrak{C}(y, R_0, R)\big) < \delta_1 R\,,$$

with δ_1 small. By applying q we get

$$\begin{aligned} d_{\mathcal{H}d}\big(q \circ \pi_0(W_1(y)) \cap \mathfrak{C}\big(q(y), LR_0, \tfrac{1}{L}R\big), q \circ \pi_0(W_2(y)) \cap \mathfrak{C}\big(q(y), LR_0, \tfrac{1}{L}R\big)\big) \\ < \delta_2 R\,. \end{aligned}$$

Let W_i' be the Weyl chambers of vertex x_0 associated to W_i by $\widetilde{q}$, $i = 1, 2$.

Let $R_{\varepsilon/4} = R_{\varepsilon/4}(R_0, L, c)$ be the radius associated by Proposition 4.2.6 to the fixed ε. By Propositions 4.2.1 and 4.2.6, and Remark 3.3.11, (2), we have

$$\widetilde{\angle}^H_{q(y)}\big(q \circ \pi_0(W_i(y)) \cap \mathfrak{C}_\xi(q(y), \rho), W_i'(q(y)) \cap \mathfrak{C}_\xi(q(y), \rho)\big) < \tfrac{\varepsilon}{4}\ \forall \rho \geq R_{\frac{\varepsilon}{4}},\ i=1, 2.$$

From the two previous inequalities we deduce that

$$d_{\mathcal{H}d}\big(W_1'(q(y)) \cap \mathfrak{C}\big(q(y), R_{\frac{\varepsilon}{4}}, \tfrac{1}{L}R\big), W_2'(q(y)) \cap \mathfrak{C}\big(q(y), R_{\frac{\varepsilon}{4}}, \tfrac{1}{L}R\big)\big) < \left(\delta_2 + \tfrac{\varepsilon}{2}\right) R \,.$$

By the convexity of the distance, in the previous relation we may replace the annulus $\mathfrak{C}\big(q(y), R_{\varepsilon/4}, \frac{1}{L}R\big)$ by the ball $B\big(q(y), \frac{1}{L}R\big)$. By (5.1), $q(y) \in N_D(H')$. Let $y' = \mathrm{proj}_{H'} q(y)$. Then $d(q(y), y') \leq D$. It follows that $W_i'(q(y))$ and $W_i'(y')$ are at Hausdorff distance D one from the other, $i = 1, 2$. This and the previous inequality imply

$$d_{\mathcal{H}d}\left(W_1'(y') \cap B\left(y', \tfrac{1}{2L}R\right), W_2'(y') \cap B\left(y', \tfrac{1}{2L}R\right)\right) \leq \left(2\delta_2 + \tfrac{\varepsilon}{2}\right) R \,. \tag{5.2}$$

The same is true if we replace y' by any other point in H'. So in the previous relation we may replace y' by $proj_{H'} x_0$. The distance from x_0 to $proj_{H'} x_0$ is at most C, by (5.1). This and the relation (5.2) modified as suggested previously imply

$$d_{\mathcal{H}d}\left(W_1' \cap B\left(x_0, \tfrac{1}{3L}R\right), W_2' \cap B\left(x_0, \tfrac{1}{3L}R\right)\right) \leq \left(3\delta_2 + \tfrac{\varepsilon}{2}\right) R \,.$$

This implies $d_{\overline{K}}(W_1', W_2') \leq \varepsilon$ if δ_2 is small enough, so if the initial δ is small enough, and the initial R big enough. □

Proposition 5.2.1 emphasizes the interest of the notion of butterfly flat. A butterfly flat F through x_0 has the property that $\tilde{q}$ is uniformly continuous on full measure sets in chamber stars of all walls in F of vertex x_0. This suggests that if we replace, in the argument of prolongation of $\tilde{q}$ to a simplicial isomorphism Φ, the set of good l.b. flats with the set of butterfly flats (or a full measure subset of it) this will make easier the proof of the continuity of Φ.

5.3 The construction of the isomorphism. Our goal is to obtain a simplicial isomorphism Φ which coincides with $\tilde{q}$ on a full measure subset of chambers. It suffices to construct a bijection on the set of chambers of $\partial_\infty X$ preserving adjacencies. We use two slightly different methods to construct Φ in the rank two case and in the higher rank case, then we show that Φ coincides with $\tilde{q}$ on a full measure set and that it is a homeomorphism. We start the construction of Φ in the same way for the two cases by fixing $k \in K$ such that kF_0 is a pistil apartment. We denote $\widetilde{\mathcal{F}} = kF_0(\infty)$ and $\widetilde{\mathcal{W}} = kW_0(\infty)$. We also denote $\widetilde{\mathcal{W}}_i = k\sigma_i W_0(\infty)$, $i \in \{1, 2, \dots, p_0\}$. We recall that $\{e, \sigma_1, \dots, \sigma_{p_0}\}$ is a system of representatives for $(K \cap N(A))/K_A$.

5.3.A The case when the rank is at least 3. In this case we use the two theorems of J. Tits [T, Chapter 4] stated below.

Let Σ be a spherical building of rank $r \geq 2$ and $\mathcal{W}$ a chamber in Σ. For every $i \in \{1, 2, \dots, r-1\}$ we denote

$$E_i(\mathcal{W}) := \{\mathcal{W}' \text{ chamber in } \Sigma \mid \mathcal{W}' \cap \mathcal{W} \text{ wall of codim} \leq i\}.$$

Let X be a symmetric space of rank $r \geq 2$ and W a Weyl chamber of vertex x_0 in X. For every $i \in \{1, 2, \dots, r-1\}$ we denote

$$E_i(W) := \{W' \text{ Weyl chamber of vertex } x_0 \text{ in } X \mid W' \cap W \text{ wall of codim} \leq i\}.$$

Theorem 5.3.1 [T, Theorem 4.1.2]**.** *Let Σ, Σ' be two spherical buildings of the same rank $r \geq 3$, $\mathcal{W} \subset \Sigma$, $\mathcal{W}' \subset \Sigma'$ two chambers and $\mathcal{A}$, $\mathcal{A}'$ two apartments containing $\mathcal{W}$ and $\mathcal{W}'$, respectively. A bijection $\phi : E_2(\mathcal{W}) \cup \mathcal{A} \to E_2(\mathcal{W}') \cup \mathcal{A}'$, $\phi(\mathcal{A}) = \mathcal{A}'$, preserving adjacencies, can be extended to an isomorphism $\Phi : \Sigma \to \Sigma'$.*

Theorem 5.3.2 [T, Theorem 4.1.1]**.** *Let Σ, Σ' be two spherical buildings of the same rank $r \geq 2$, $\mathcal{W}$ a chamber in Σ and $\mathcal{A}$ an apartment containing $\mathcal{W}$. If two isomorphisms $\Phi_1 : \Sigma \to \Sigma'$ and $\Phi_2 : \Sigma \to \Sigma'$ coincide on $E_1(\mathcal{W}) \cup \mathcal{A}$ then they coincide.*

In [T, Chapter 4], J. Tits gives the proofs of the two theorems in the irreducible case, but the proofs also work in the general case.

Since kF_0 is a pistil flat, for every wall $\mathcal{M}_0$ in F_0, $\mathbf{L} \cap St(k\mathcal{M}_0)$ has full measure in $St\,(k\mathcal{M}_0)$. The map $\tilde{q}$ is uniformly continuous on $\mathbf{L} \cap St(k\mathcal{M}_0)$, so it has a unique prolongation to $St\,(k\mathcal{M}_0)$. Thus, $\tilde{q}$ induces a unique adjacence preserving bijection $\phi : E_2(\widetilde{\mathcal{W}}) \cup \widetilde{\mathcal{F}} \to E_2(\widetilde{\mathcal{W}}') \cup \widetilde{\mathcal{F}}'$, where $\widetilde{\mathcal{W}}'$ and $\widetilde{\mathcal{F}}'$ are the horizons of $q(\pi_0(kW_0))$ and $q(\pi_0(kF_0))$, respectively. By Theorem 5.3.1, ϕ prolongates to an isomorphism $\Phi : \partial_\infty X \to \partial_\infty X$. Since ϕ is uniformly continuous on every chamber star of every panel of $\widetilde{\mathcal{W}}$, the same follows for Φ.

Similarly, $\tilde{q}$ induces an adjacence preserving bijection ϕ_i defined on $E_2(\widetilde{\mathcal{W}}_i) \cup \widetilde{\mathcal{F}}$, $i \in \{1, 2, \dots, p_0\}$, which prolongates to an isomorphism Φ_i. We show by induction on d that if $\widetilde{\mathcal{W}}_i$ is at combinatorial distance d from $\widetilde{\mathcal{W}}$, then $\Phi_i = \Phi$. If $d = 2$, then $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}_i$ are adjacent, so $E_1(\widetilde{\mathcal{W}}) \subset E_2(\widetilde{\mathcal{W}}) \cap E_2(\widetilde{\mathcal{W}}_i)$. The restrictions $\phi|_{E_1(\widetilde{\mathcal{W}})}$ and $\phi_i|_{E_1(\widetilde{\mathcal{W}})}$ both coincide with the unique prolongation of $\tilde{q}|_{\mathbf{L} \cap E_1(kW_0)}$. Also, ϕ and ϕ_i coincide on $\widetilde{\mathcal{F}}$. By Theorem 5.3.2, $\Phi = \Phi_i$.

If $\widetilde{\mathcal{W}}_i$ is at combinatorial distance $d+1$ from $\widetilde{\mathcal{W}}$, we consider the last-but-one element, $\widetilde{\mathcal{W}}_j$, of a gallery stretched from $\widetilde{\mathcal{W}}$ to $\widetilde{\mathcal{W}}_i$. By induction $\Phi = \Phi_j$. We prove $\Phi_j = \Phi_i$ with an argument similar to the one in the initial step of the induction.

We show that Φ coincides with $\tilde{q}$ on a full measure subset of $\overline{K}$. By Lemma 2.5.1, $\overline{K} = \cup_{i=0}^{p_0} \Omega_{m_i}(k\sigma_i)$, where $m_0, m_1, m_2, \dots, m_{p_0}$ are positive real numbers. It suffices to show Φ coincides with $\tilde{q}$ on a full measure subset of each $\Omega_{m_i}(k\sigma_i)$, $i \in \{0, 1, 2, \dots, p_0\}$. So in the sequel i is fixed. We denote $\varpi_i = op_{\widetilde{\mathcal{F}}} \circ \mathrm{retr}_{\widetilde{\mathcal{F}}, \widetilde{\mathcal{W}}_i}$.

We prove by induction on ℓ that for every chamber $\mathcal{W}$ at combinatorial distance ℓ from $\widetilde{\mathcal{W}}_i$ and tied to $\widetilde{\mathcal{F}}$ with respect to $\widetilde{\mathcal{W}}_i$, $\Phi(\mathcal{W}) = \tilde{q}(\mathcal{W})$. The statement is true for $\ell = 1$ and $\ell = 2$. We suppose it is true for ℓ. Let $\mathcal{W}$ be a chamber at combinatorial distance $\ell+1$ from $\widetilde{\mathcal{W}}_i$, tied to $\widetilde{\mathcal{F}}$ with respect to $\widetilde{\mathcal{W}}_i$. Then $\mathcal{W}$ and $\varpi_i(\mathcal{W})$ determine a butterfly apartment, $\mathcal{F}_{\mathcal{W}}$. Let $\mathfrak{y}$ be a label in the set $\{1, 2, \dots r\}$ and $\mathcal{W}_{\mathfrak{y}}$ be a chamber $\mathfrak{y}$-adjacent to $\mathcal{W}$, contained in a minimal gallery from $\widetilde{\mathcal{W}}_i$ to $\mathcal{W}$. In $\mathcal{F}_{\mathcal{W}}$ we consider the chamber $\mathcal{W}'_{\mathfrak{y}}$ which is $op(\mathfrak{y})$-adjacent to $\varpi_i(\mathcal{W})$. We notice that $\mathcal{W}'_{\mathfrak{y}}$ is opposite to $\mathcal{W}_{\mathfrak{y}}$. We have that $\Phi(\mathcal{W}_{\mathfrak{y}}) = \tilde{q}(\mathcal{W}_{\mathfrak{y}})$ by induction and $\Phi(\mathcal{W}'_{\mathfrak{y}}) = \tilde{q}(\mathcal{W}'_{\mathfrak{y}})$ since Φ coincides with $\tilde{q}$ on a full measure set of every chamber star of every panel in $\widetilde{\mathcal{F}}$. Then $\Phi|_{\mathcal{F}_{\mathcal{W}}} \equiv \tilde{q}|_{\mathcal{F}_{\mathcal{W}}}$ and $\Phi(\mathcal{W}) = \tilde{q}(\mathcal{W})$.

For ℓ equal to the combinatorial diameter of $\partial_\infty X$, we obtain that Φ coincides with $\tilde{q}$ on a full measure subset of $\Omega(k\sigma_i)$, so on a full measure subset of $\Omega_{m_i}(k\sigma_i)$.

5.3.B The rank two case We start with the adjacence preserving bijection $\psi : \cup_{i=0}^{p_0} E_1(\widetilde{\mathcal{W}}_i) \to \cup_{i=0}^{p_0} E_1(\widetilde{\mathcal{W}'_i})$ induced by $\tilde{q}$, where $\widetilde{\mathcal{W}'_i}$ is the horizon of $q(\pi_0(k\sigma_i W_0))$, $\cup_{i=0}^{p_0} \widetilde{\mathcal{W}'_i} = \tilde{q}(\widetilde{\mathcal{F}})$ and $\psi(\widetilde{\mathcal{F}}) = \tilde{q}(\widetilde{\mathcal{F}})$. Up to an isometric change of labelling we may suppose that ψ preserves labelling, by Lemma 2.2.1. The map ψ is uniformly continuous on the chamber star of each panel in $\widetilde{\mathcal{F}}$. We prolongate ψ to an isomorphism.

We denote $q_0 = \frac{p_0+1}{2}$ the combinatorial diameter of a half-apartment, $Ch(m)$ the set of chambers in $\partial_\infty X$ at combinatorial distance m from $\widetilde{\mathcal{W}}$ and ϖ the map $op_{\widetilde{\mathcal{F}}} \circ \mathrm{retr}_{\widetilde{\mathcal{F}}, \widetilde{\mathcal{W}}}$. We prove by induction on $m \in \mathbb{N} \cap [1, q_0]$ the following:

(Pro_m) There exists an adjacence preserving bijection ψ_m defined on $\cup_{i=1}^m Ch(i)$ such that ψ_m coincides with ψ on the intersection of their domains of definition, and for every $i \in \{1, 2, \dots, m\}$, ψ_m is continuous on $Ch(i)$ and coincides with $\tilde{q}$ on the set of chambers tied to $\widetilde{\mathcal{F}}$ with respect to $\widetilde{\mathcal{W}}$.

(Pro_2) is obviously true, $\psi_2 = \psi|_{E_1(\widetilde{\mathcal{W}})}$. We suppose ($\mathrm{Pro}_m$) is true and prove ($\mathrm{Pro}_{m+1}$), $m + 1 \leq q_0$. We want to prolongate ψ_m to ψ_{m+1}. In order to define ψ_{m+1} on $Ch(m + 1)$ we use the bijection ψ. For every chamber $\mathcal{W}$ in $Ch(m+1)$, we take the last-but-one chamber, $\mathcal{W}' \in Ch(m)$, of a minimal gallery from $\widetilde{\mathcal{W}}$ to $\mathcal{W}$. Suppose $\mathcal{W}'$ and $\mathcal{W}$ are $\mathfrak{y}$-adjacent, $\mathfrak{y} \in \{1,2\}$. In the apartment containing $\mathcal{W}$ and $\varpi(\mathcal{W})$, $\mathcal{W}'$ is opposite to the chamber $\mathcal{W}''$, $op(\mathfrak{y})$-adjacent to $\varpi(\mathcal{W})$. The chambers $\psi_m(\mathcal{W}')$ and $\psi(\mathcal{W}'')$ are opposite by Lemma 2.2.3 (iii), so they define an unique apartment. We take as $\psi_{m+1}(\mathcal{W})$ the chamber in this new apartment $\mathfrak{y}$-adjacent to $\psi_m(\mathcal{W}')$. The continuity of ψ_m on $Ch(m)$ and of ψ on the chamber stars of panels in $\widetilde{\mathcal{F}}$ imply the continuity of ψ_{m+1} on $Ch(m+1)$.

The properties of ψ_{m+1} of being adjacence preserving and of coinciding with ψ on the intersection of their domains are obvious from the construction. Since $\widetilde{\mathcal{F}}$ is a pistil apartment, a.e. chamber $\mathcal{W} \in Ch(m+1)$ is tied to $\widetilde{\mathcal{F}}$ with respect to $\widetilde{\mathcal{W}}$. By induction and with the previous notations, for such a chamber $\mathcal{W}$, $\psi_m(\mathcal{W}') = \tilde{q}(\mathcal{W}')$, and also $\psi(\mathcal{W}'') = \tilde{q}(\mathcal{W}'')$. We may conclude that $\psi_{m+1}(\mathcal{W}) = \tilde{q}(\mathcal{W})$.

We obtain in the end $\Psi_0 = \psi_{q_0}$. In order to prolongate Ψ_0 to the set $Ch(q_0+1)$ of chambers opposite to $\widetilde{\mathcal{W}}$, we can consider, for each opposite chamber, one of the two galleries joining it to $\widetilde{\mathcal{W}}$. If, for each opposite chamber, we consider only the gallery from $\widetilde{\mathcal{W}}$ to it containing a chamber $\mathfrak{y}$-adjacent to $\widetilde{\mathcal{W}}$, with the previous argument we can prolongate Ψ_0 to $\Psi_{\mathfrak{y}}$, $\mathfrak{y} = 1, 2$. Both Ψ_1 and Ψ_2 are bijections continuous on $Ch(q_0+1)$ and coincide with $\tilde{q}$ on almost every chamber of $Ch(q_0+1)$. Then $\Psi_1 = \Psi_2 = \Psi$, and Ψ is an adjacence preserving bijection on the set of all chambers, so it induces an isomorphism Φ on $\partial_\infty X$. This isomorphism has the properties that it coincides with ψ on the domain of definition of ψ and that it coincides with $\tilde{q}$ on a full measure subset of $\Omega(k)$. An identical reasoning can be done to prolongate the map ψ starting with the chamber $\widetilde{\mathcal{W}_i}$ instead of $\widetilde{\mathcal{W}}$. We would obtain an isomorphism Φ_i which coincides with $\tilde{q}$ on a full measure subset of $\Omega(k\sigma_i)$. By Theorem 5.3.2 and the fact that Φ_i and Φ coincide with ψ on its domain of definition, $\Phi_i = \Phi$, $\forall i \in \{1, 2, \dots p_0\}$. Then Φ also coincides with $\tilde{q}$ on a full measure subset of $\Omega(k\sigma_i)$, $\forall i \in \{1, 2, \dots p_0\}$. Since $\overline{K} = \cup_{i=0}^{p_0} \Omega(k\sigma_i)$, Φ coincides with $\tilde{q}$ on a full measure subset of $\overline{K}$.

5.3.C The continuity. We have obtained the simplicial isomorphism $\Phi : \partial_\infty X \to \partial_\infty X$. We prove the following general result which implies that Φ is a homeomorphism in the cone topology.

LEMMA 5.3.3. *Let $\Phi : \partial_\infty X \to \partial_\infty X$ be a simplicial isomorphism and $\widetilde{\mathcal{F}}$ an apartment in $\partial_\infty X$. If Φ is continuous on the chamber star of each panel in $\widetilde{\mathcal{F}}$, then Φ is a homeomorphism in the cone topology.*

Proof. Up to an isometric change of labelling on the target building, we may suppose that Φ preserves labelling (easy consequence of [T, §2.6]). It suffices then to show that Φ is an homeomorphism on $Ch\,(\partial_\infty X)$. Thus, in the rest of the proof we shall denote by Φ the restriction of Φ to $Ch\,(\partial_\infty X)$. Since Φ is a bijection between compact sets , in order to show that Φ is an homeomorphism it suffices to show its continuity.

Let $g \in G$ such that $\widetilde{\mathcal{F}} = g\mathcal{F}_0$. By Lemma 2.5.1, $\overline{K} = \cup_{i=0}^{p_0}\Omega_{m_i}(g\sigma_i)$, where $m_i \in \mathbb{R}_+^*$, $\forall i \in \{0, 1, \dots p_0\}$. Hence, it suffices to prove that Φ is continuous on each $\Omega_{m_i}(g\sigma_i)$. We fix $i \in \{0, 1, \dots p_0\}$ and we denote $g\sigma_i\mathcal{W}_0$ by $\widetilde{\mathcal{W}_i}$ and $\Omega_{m_i}(g\sigma_i)$ by Ω_i. For any $\overline{k^{op}}$ in Ω_i there exists a unique apartment $\mathcal{F}(k^{op})$ containing $\widetilde{\mathcal{W}_i}$ and $k^{op}\mathcal{W}_0$. We denote the set of all chambers of all apartments $\mathcal{F}(k^{op})$, $\overline{k^{op}} \in \Omega_i$, by Ch_i. We consider the retraction $\mathrm{retr} = \mathrm{retr}_{\widetilde{\mathcal{F}},\widetilde{\mathcal{W}_i}}$ and $\varpi_i = op_{\widetilde{\mathcal{F}}} \circ \mathrm{retr}$. We prove by induction on ℓ the following statement:

($\mathbf{P}_\ell$) Let $\mathcal{W}_\lambda$ be a chamber and $(\mathcal{W}_n)$ be a sequence of chambers in Ch_i at combinatorial distance ℓ from $\widetilde{\mathcal{W}_i}$, with $\mathrm{retr}(\mathcal{W}_\lambda) = \mathrm{retr}(\mathcal{W}_n)$, $\forall n \in \mathbb{N}$. Then, the following implication holds

$$\mathcal{W}_n \to \mathcal{W}_\lambda \quad \Rightarrow \Phi(\mathcal{W}_n) \to \Phi(\mathcal{W}_\lambda)\,.$$

We prove ($\mathbf{P_2}$). In this case $\mathcal{W}_n$ and $\mathcal{W}_\lambda$ have a panel in common with $\widetilde{\mathcal{W}_i}$ and Φ is continuous on the chamber star of this panel.

We suppose ($\mathbf{P}_\ell$) is true and prove ($\mathbf{P}_{\ell+1}$). Let $\mathcal{W}_\lambda, \mathcal{W}_n$ be chambers at combinatorial distance $\ell+1$ from $\widetilde{\mathcal{W}_i}$. Since $\mathrm{retr}(\mathcal{W}_\lambda) = \mathrm{retr}(\mathcal{W}_n)$, $\forall n \in \mathbb{N}$, there exists a label $\mathfrak{y} \in \{1, 2, \dots, r\}$ and a chamber $\mathcal{W}_s^{\mathfrak{y}}$ $\mathfrak{y}$-adjacent to $\mathcal{W}_s$ and contained in a minimal gallery from $\widetilde{\mathcal{W}_i}$ to $\mathcal{W}_s$, $\forall s \in \{\lambda\} \cup \mathbb{N}$. Also $\mathrm{retr}(\mathcal{W}_\lambda^{\mathfrak{y}}) = \mathrm{retr}(\mathcal{W}_n^{\mathfrak{y}})$, $\forall n \in \mathbb{N}$, and $\mathcal{W}_s^{\mathfrak{y}}$ are at combinatorial distance ℓ from $\widetilde{\mathcal{W}_i}$.

(A) $\mathcal{W}_n \to \mathcal{W}_\lambda$ implies $\mathcal{W}_n^{\mathfrak{y}} \to \mathcal{W}_\lambda^{\mathfrak{y}}$.

Otherwise, by eventually taking a subsequence, we would have $\mathcal{W}_n^{\mathfrak{y}} \to \mathcal{W}' \neq \mathcal{W}_\lambda^{\mathfrak{y}}$. Since $\mathcal{W}_s \in Ch_i$, each $\mathcal{W}_s$ is contained, together with $\widetilde{\mathcal{W}_i}$, in the boundary of a maximal flat F_s with $d(x_0, F_s) < m_i$, $\forall s \in \{\lambda\} \cup \mathbb{N}$. By eventually taking a subsequence, we may suppose that F_n converges in the modified Hausdorff metric to a flat F with $d(x_0, F) \leq m_i$. Also, $F(\infty)$ contains $\widetilde{\mathcal{W}_i}$ and $\mathcal{W}_\lambda$, so it contains also $\mathcal{W}_\lambda^{\mathfrak{y}}$, which is the unique chamber $\mathfrak{y}$-adjacent to $\mathcal{W}_\lambda$ and closer than it to $\widetilde{\mathcal{W}_i}$. The chamber $\mathcal{W}_n^{\mathfrak{y}}$ has the same properties in the boundary $F_n(\infty)$ with respect to $\widetilde{\mathcal{W}_i}$ and $\mathcal{W}_n$. This and the fact that $F_n \to F$ imply that $\mathcal{W}_n^{\mathfrak{y}} \to \mathcal{W}_\lambda^{\mathfrak{y}}$. We have obtained a contradiction.

(B) Let $\mathcal{W}^{op} = \varpi_i(\mathcal{W}_s)$ which is opposite to $\mathcal{W}_s$, $\forall s \in \{\lambda\} \cup \mathbb{N}$. The unique apartment $\mathcal{F}_s$ determined by $\mathcal{W}^{op}$ and $\mathcal{W}_s$ contains $\widetilde{\mathcal{W}_i}$. For each $s \in \{\lambda\} \cup \mathbb{N}$, let $(\mathcal{W}'_s)^{\mathfrak{y}}$ be the unique chamber of $\mathcal{F}_s$ $op(\mathfrak{y})$-adjacent to $\mathcal{W}^{op}$. We notice that $\mathcal{W}_s^{\mathfrak{y}}$ and $(\mathcal{W}'_s)^{\mathfrak{y}}$ are opposite. With an argument analogous to the one in **(A)** we may show that $(\mathcal{W}'_n)^{\mathfrak{y}} \to (\mathcal{W}'_\lambda)^{\mathfrak{y}}$.

(C) By $(\mathbf{P}_\ell)$, $\Phi\left(\mathcal{W}_n^{\mathfrak{y}}\right) \to \Phi\left(\mathcal{W}_\lambda^{\mathfrak{y}}\right)$, and by hypothesis $\Phi\left((\mathcal{W}'_n)^{\mathfrak{y}}\right) \to \Phi\left((\mathcal{W}'_\lambda)^{\mathfrak{y}}\right)$. Since $\Phi\left(\mathcal{W}_s^{\mathfrak{y}}\right)$ and $\Phi\left((\mathcal{W}'_s)^{\mathfrak{y}}\right)$ are opposite, contained in the apartment $\Phi(\mathcal{F}_s)$, then $\Phi(\mathcal{F}_n) \to \Phi(\mathcal{F}_\lambda)$, which implies $\Phi(\mathcal{W}_n) \to \Phi(\mathcal{W}_\lambda)$. $\square$

5.4 The associated isometry. There is a unique isometry g in G having Φ as its boundary map, provided G doesn't have a rank one factor. To prove that g is at a finite distance from q, we need

LEMMA 5.4.1. *Let $\Delta_0 = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be a fundamental set of roots and $k_i \in K_{\alpha_i}$ such that $d(\bar{e}, \bar{k}_i) = \pi/2$, $\forall i \in \{1, 2, \dots, r\}$. Then for every $C > 0$ there exists $D = D(C)$ such that*

$$N_C(F_0) \cap \bigcap_{i=1}^{r} N_C(k_i F_0) \subset N_D(x_0)\,.$$

Proof. Suppose there exist sequences $k_{in} \in K_{\alpha_i}$ such that $d(\bar{e}, \bar{k}_{in}) = \pi/2$, $\forall i \in \{1, 2, \dots, r\}$, and $z_n \in N_C(F_0) \cap \cap_{i=1}^r N_C(k_{in} F_0)$ with $d(z_n, x_0) = \iota_n \to \infty$. Then in $Con_\omega(X, x_0, (\iota_n))$ the limit flats $F_\omega^0 = [F_0]$ and $F_\omega^i = [k_{in} F_0]$ have the property that their intersection is a point, $x_\omega^0 = [x_0]$. On the other hand their intersection also contains the point $z_\omega = [z_n]$ which is at distance 1 from x_ω^0. We get a contradiction. $\square$

We are now in a position to prove

PROPOSITION 5.4.2. *There exists a constant $D = D(L, c, \Gamma)$ such that*

$$d(q(x), g(x)) \le D\,, \quad \forall x \in \Gamma x_0.$$

Proof. STEP 1. First we prove that in any point z of the orbit Γx_0 a big set of flats through z are good R_0-l.b. flats and their chambers in $\partial_\infty X$ are in the set of chambers on which Φ coincides with $\tilde{q}$. To any of these flats q associates a fan over an apartment, so a unique flat. By Proposition 4.2.7, the point $q(z)$ is uniformly close to each of these image flats.

We consider the set $Log(R_0)$ defined in Lemma 3.2.2, of measure at least $1 - \frac{s}{log R_0}$. We denote $\frac{2s}{\log R_0}$ by $\varkappa_0$. By Fubini theorem, the set of points $\bar{x} \in \mathcal{V}$ with the property that, if F_x is a maximal flat through x, then

$$\mu_{K_x}\left(\{k \in K_x \mid kF_x \text{ is } R_0\text{-l.b. with respect to } x\}\right) \ge 1 - \varkappa_0\,,$$

has measure at least $1/2$ in $\mathcal{V}$. We have chosen x_0 in a full measure subset. So we may moreover suppose that x_0 is chosen so that $\bar{x}_0$ is in the previous subset of $\mathcal{V}$ of measure greater than $1/2$. Then

$$\mu_K\big(\mathbf{L}_K \cap \{k \in K \mid kF_0 \text{ is } R_0- \text{ l.b. with respect to } x_0\}\big) \geq 1 - \varkappa_0 \,.$$

Let $z = \gamma x_0$ be a point in Γx_0. Let $\overline{K}_z = K_z/(K_z \cap Z(\gamma A \gamma^{-1}))$ and $\pi_z : K_z \to \overline{K}_z$. Let F_z denote γF_0, and let $\{e, \tau_1, \tau_2, \dots, \tau_{p_0}\}$ be a system of representatives for $N(\gamma A \gamma^{-1}) \cap K_z/(Z(\gamma A\gamma^{-1}) \cap K_z)$. If kF_0 is a R_0–l.b. or a good l.b. flat with respect to x_0 then $\gamma k F_0 = \gamma k \gamma^{-1} F_z$ is a R_0–l.b. or a good l.b. flat with respect to z. Consequently we have $\mu_{K_z}(\mathfrak{G}_z) \geq 1 - \varkappa_0$, where

$$\mathfrak{G}_z := \{k \in K_z \mid kF_z \text{ is a good } R_0\text{–l.b. flat with respect to } z\}\,.$$

The map Φ restricted to $Ch\,(\partial_\infty X)$, which we identify with $\overline{K}$, coincides with $\tilde{q}$ on a full measure set. We denote this set by Θ. The map $\phi_{x_0 z} : \overline{K} \to \overline{K}_z$, $\phi_{x_0 z}(\bar{k}) = \bar{k}'$ such that $k'\gamma W_0$ is the Weyl chamber of vertex z asymptotic to kW_0, is a diffeomorphism. So $\Theta_z = \phi_{x_0 z}(\Theta)$ has full measure in $\overline{K}_z$. It follows that $\Theta_{K_z} = \pi_z^{-1}(\Theta_z)$ has full measure in K_z. The set

$$\mathfrak{G}'_z = \mathfrak{G}_z \cap \Theta_{K_z} \cap \bigcap_{i=1}^{p_0} \Theta_{K_z} \tau_i^{-1}$$

has measure at least $1 - \varkappa_0$. Let $k \in \mathfrak{G}'_z$. The flat kF_z is a good R_0–l.b. flat with respect to z, and all its Weyl chambers are in the set Θ_z. The quasi-isometry q associates to kF_z a fan, $\cup_{i=0}^{p_0} W_i$, of vertex $q(z)$ over an apartment, and if F' is the maximal flat asymptotic to it, then $d(q(z), F') \leq C$, where $C = C(R_0, L, c)$ (Proposition 4.2.7). Also $\tilde{q}(kF_z(\infty)) = \cup_{i=0}^{p_0} W_i(\infty)$, so the same thing is true for Φ, which implies that $g(kF_z) = F'$. Hence

$$d\big(q(z), g(kF_z)\big) \leq C\,. \tag{5.3}$$

Step 2. If $\mathfrak{G}'_z$ is big enough ($\varkappa_0$ is small enough), we will be able to choose a finite family of flats in it intersecting a fixed flat in r linearly independent hyperplanes. This, Lemma 5.4.1 and the inequality (5.3) will imply the conclusion.

For every $\alpha \in \Delta_0$ we fix $k_\alpha = \gamma k'_\alpha \gamma^{-1}$, where $k'_\alpha \in K_\alpha$ is such that $d_{\overline{K}_\alpha}(\bar{e}, \bar{k}'_\alpha) = \pi/2$. The set

$$\mathfrak{G}''_z = \mathfrak{G}'_z \cap \Big(\bigcap_{\alpha \in \Delta_0} \mathfrak{G}'_z k_\alpha^{-1} \Big)$$

has measure at least $1 - (r+1)\varkappa_0$. So for $\varkappa_0$ small enough $\mathfrak{G}''_z \neq \emptyset$. We consider an element $k \in \mathfrak{G}''_z$. Then k and kk_α, $\alpha \in \Delta_0$, are in $\mathfrak{G}'_z$. By (5.3) we then have

$$d(q(z), g(kF_z)) \le C \ \text{ and } \ d(q(z), g(kk_\alpha F_z)) \le C, \forall \alpha \in \Delta_0 . \tag{5.4}$$

Also, $kF_z \cap \bigcap_{\alpha\in\Delta_0} kk_\alpha F_z = \{z\}$. Since $kF_z = k\gamma F_0$ and $kk_\alpha F_z = k\gamma k'_\alpha F_0$, with $d_{\overline{K}_\alpha}(\bar{e}, \bar{k}'_\alpha) = \frac{\pi}{2}$, by Lemma 5.4.1, there exists $D = D(R_0, L, c)$ such that

$$N_C(kF_z) \cap \bigcap_{\alpha\in\Delta_0} N_C(kk_\alpha F_z) \subset N_D(z) . \tag{5.5}$$

If $F' = g(kF_z)$ and $F'_\alpha = g(kk_\alpha F_z)$, since g is an isometry we have

$$F' \cap \bigcap_{\alpha\in\Delta_0} F'_\alpha = \{gz\} \text{ and } N_C(F') \cap \bigcap_{\alpha\in\Delta_0} N_C(F'_\alpha) \subset N_D(gz) . \tag{5.6}$$

By (5.4) and (5.6), we may conclude that

$$q(z) \in N_C(F') \cap \bigcap_{\alpha\in\Delta_0} N_C(F'_\alpha) \subset N_D(gz) .$$

Thus $d(q(z), gz) \le D$. □

In particular Proposition 5.4.2 says that the isometry g sends the entire orbit Γx_0 in a neighborhood of itself $N_D(\Gamma x_0)$. We have the following result due to Nimish Shah.

Theorem 5.4.3 [Sh]**.** *Let G be a semisimple Lie group without compact factors and Γ an irreducible non-uniform lattice in G. If $g \in G$ has the property that $g\Gamma \subset N_D(\Gamma)$, for a certain D, then $g \in Comm(\Gamma)$.*

This and the previous proposition imply Theorem 1.2. □

Corollary 5.4.4. *Let Γ be a non-uniform irreducible lattice in a semisimple group G of rank at least 2, with finite center and without factors of rank ≤ 1. The group $QI(\Gamma)$ of quasi-isometries of Γ coincides with the commensurator of Γ.*

Proof of Theorem 1.1. (1) Every $\lambda \in \Lambda$ defines a quasi-isometry on Γ by $q \circ \lambda \circ q^{-1}$. By Theorem 1.2 we can associate to it an isometry $\phi(\lambda) \in Comm(\Gamma)$. The proof of the fact that $\phi : \Lambda \to Comm(\Gamma)$ is a homomorphism with finite kernel and discrete image is identical to the one given by R. Schwartz [S1, 10.4].

In order to prove that $\phi(\Lambda) = \Gamma_1$ is a non-uniform lattice it suffices to notice that Γ is at a bounded distance from Γ_1: for every $\gamma \in \Gamma$, $\phi(q^{-1}(\gamma))$ is at a finite distance from γ.

(2) Suppose Γ is irreducible. We may also suppose, up to finite index, that Γ_1 is without torsion. Then the fact that Γ_1 is quasi-isometric to Γ and the arguments we used to prove (1) imply that there is an isomorphism ϕ

from Γ_1 to a non-uniform lattice $\Gamma_1' \subset Comm(\Gamma)$ which is at a finite distance from Γ. By Mostow rigidity [Mos], Γ_1 and Γ_1' are conjugate. The following theorem of N. Shah allows to conclude that $\Gamma_1' \cap \Gamma$ has finite index in Γ_1' and in Γ. □

Theorem 5.4.5 [Sh, Corollary 1.5]**.** *Let G be a connected semisimple group without compact factors. Let Γ and Γ_1 be two lattices, Γ non-uniform irreducible. If for some $g \in G$, the projection of $g\Gamma_1$ on $\Gamma\backslash G$ is contained in a compact subset, then $g\Gamma_1 g^{-1} \cap \Gamma$ is a subgroup of finite index in $g\Gamma_1 g^{-1}$ as well as in Γ.*

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