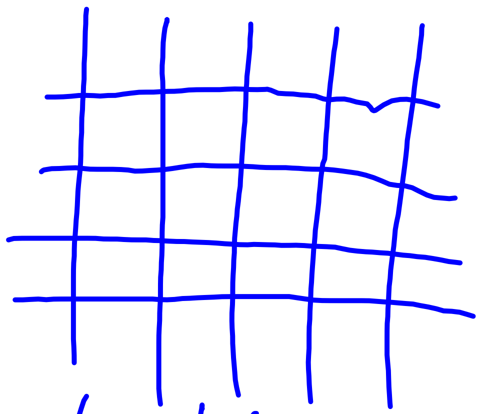
Ex. ① $G = \mathbb{Z}^2$ $\underset{=1=}{S} = \{(\pm 1, 0), (0, \pm 1)\}$

$Cay(G, S)$

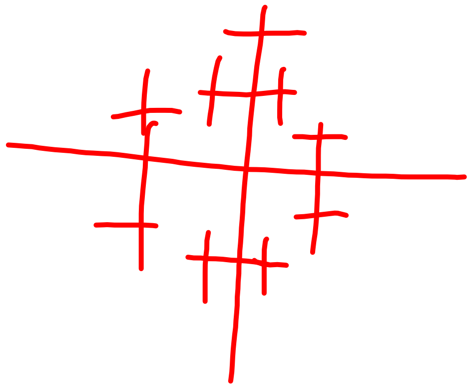


② $G = F_2$ $S = \{a, b, a^{-1}, b^{-1}\}$

$\shortparallel$

reduced words: $\nexists$ xx^{-1}. + empty

$Cay(F_2, S)$

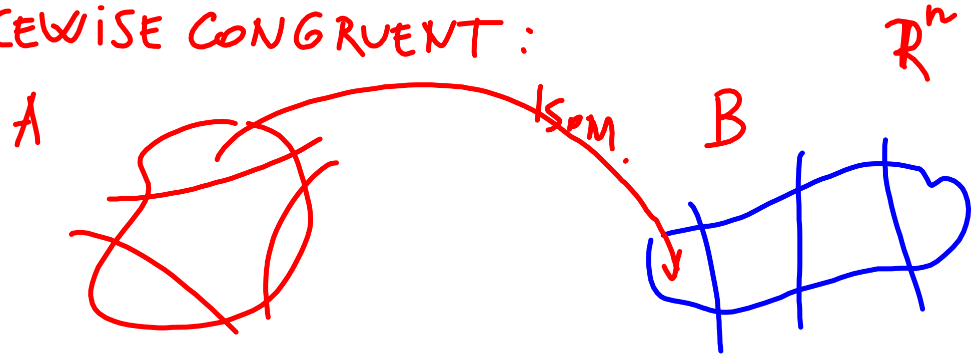


=2=

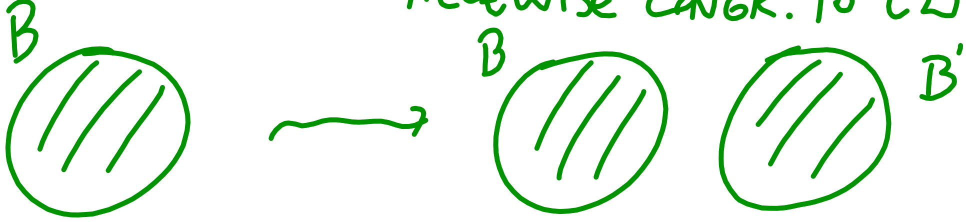
$G \curvearrowright M$ RIEM. MAN.

LINEAR: $G \leq GL(n,F)$

PIECEWISE CONGRUENT:



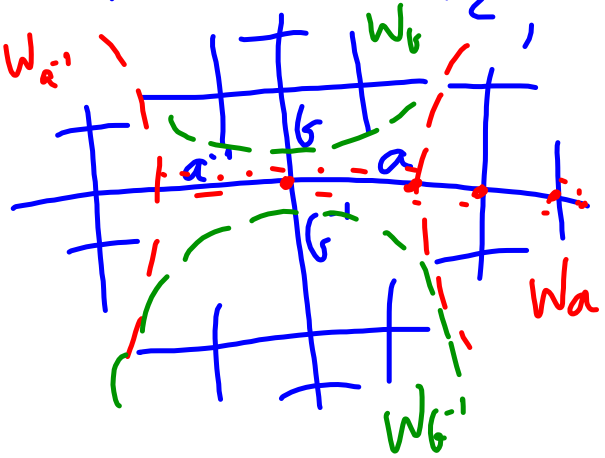
E PARADOXICAL = E PIECEWISE CONGR. TO $E \sqcup E$



= 3 =

$F_2 = F(\underbrace{\{a, b, a^{-1}, b^{-1}\}}_{\mathcal{A}})$ = set of all reduced words w in $\mathcal{A}$ + $w_\emptyset$

$\forall$ u word in F_2, $W_u := \{ w$ with prefix $u \}$.



$$F_2 = W_a \sqcup W_{a^{-1}} \sqcup W_b \sqcup W_{b^{-1}} \sqcup \{1\}$$

$$W_a \sqcup L_a W_{a^{-1}} = F_2$$

$$W_b \sqcup L_b W_{b^{-1}} = F_2$$

Modify: $W_a' = W_a \setminus \{ a^n ; n \geq 1 \}$,

$W_{a^{-1}}' = W_{a^{-1}} \sqcup \{ a^n ; n \geq 0 \}$

$F_2 = W_a' \sqcup W_{a^{-1}}' \sqcup W_b \sqcup W_{b^{-1}}$; $W_a' \sqcup L_a W_{a^{-1}}' = F_2$

$W_b \sqcup L_b W_{b^{-1}} = F_2$

= 4 =

STEP 2 USING AC, PICK $\Delta \subseteq X$ S.T. Δ INTERSECTS EVERY F_2-ORBIT EXACTLY ONCE.

ASSUME LEFT ACTION.

$$X = \bigsqcup_{g \in F_2} g\Delta .$$

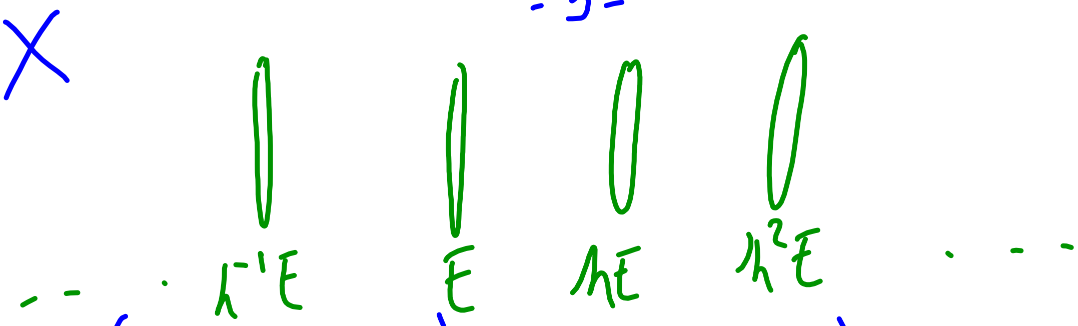
TAKE $X = W_a' \Delta \sqcup W_{a^{-1}}' \Delta \sqcup W_b \Delta \sqcup W_{b^{-1}} \Delta.$

HERE, FOR $B \leq F_2$, $B\Delta := \bigsqcup_{b \in B} b\Delta.$

$$W_a' \Delta \sqcup a W_{a^{-1}}' \Delta = F_2 \Delta = X .$$

$$W_b \Delta \sqcup b W_{b^{-1}} \Delta = F_2 \Delta = X .$$

=5=



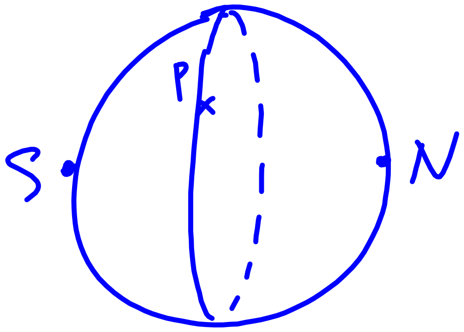
$$X = \Big(X \setminus \bigsqcup_{k\geq 0} h^k E\Big) \sqcup \Big(\bigsqcup_{k\geq 0} h^k E\Big)$$

$$\downarrow \mathrm{id} \qquad\qquad \downarrow h$$

$$X\setminus E = \Big(X \setminus \bigsqcup_{k\geq 0} h^k E\Big) \sqcup \bigsqcup_{k\geq 1} h^k E$$

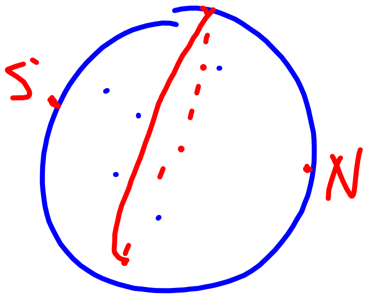
=6=

① S^2



Take h irrational rotation (no finite orbit) ⇒ never have $h^k p = p$.

②



=7=

STEP 4 (HAUSDORFF PARADOX):

- $F_2 \leq SO(3)$: TAKE $A, B \in SO(3)$,

A WITH ALGEBRAIC ENTRIES, B WITH A TRANSCENDENTAL ENTRY.

- EVERY REDUCED WORD $\underset{\neq W_\emptyset}{W(A,B)}$ IS NEVER Id_3. (A OF ∞ ORDER)

EX. OTHER EXAMPLES (PING-PONG).

ALL F_2 ARE ROTATIONS, $\forall R \in F_2$, $Fix(R)$ HAS CARDINALITY ≤ 2.

NOW TAKE $S^2 \setminus \overbrace{\bigcup_{R \in F_2} Fix(R)}^{C} = X$.

-8-

STEP 5 • WE OBTAINED S^2 IS $SO(3)$-PARADOXICAL.

• $B^3 \setminus \{0\} = \bigsqcup_{\lambda \in (0,1]} \lambda S^2$, WHERE

$$\lambda S^2 = \{ \bar{x} \, ; \, |\bar{x}| = \lambda \}.$$

WE HAD $S^2 = A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2$ S.T.

$\exists \, g, h \in SO(3)$ s.t. $A_1 \sqcup g A_2 = S^2$

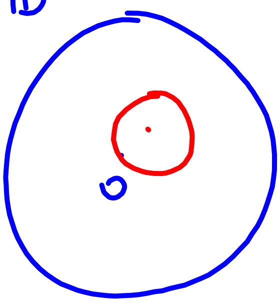
$B_1 \sqcup h B_2 = S^2$

TAKE $B^3 \setminus \{0\} = \bigsqcup_{\lambda \in (0,1]} \lambda A_1 \sqcup \bigsqcup_{\lambda \in (0,1]} \lambda A_2 \sqcup \bigsqcup_{\lambda} \lambda B_1 \sqcup \bigsqcup \lambda B_2.$

= 9 =

- $\mathbb{B}^3 \setminus \{0\}$ is $\underbrace{\text{PIECEWISE CONGRUENT}}_{\text{WRTO } \mathrm{ISOM}(\mathbb{R}^3)}$ TO $\mathbb{B}^3$.

$\mathbb{B}^3$



PICK $\Sigma \subseteq \mathbb{B}^3$ SMALL SPHERE, $0 \in \Sigma$.

$$\mathbb{B}^3 = (\mathbb{B}^3 \setminus \Sigma) \sqcup \Sigma$$

$$\mathbb{B}^3 \setminus \{0\} = (\mathbb{B}^3 \setminus \Sigma) \sqcup (\Sigma \setminus \{0\})$$

WE PROVED Σ AND $\Sigma \setminus \{0\}$ ARE PIECEWISE CONGRUENT.

(WRTO $\mathrm{STAB}_{\mathrm{ISOM}(\mathbb{R}^n)}(\mathrm{CENTER}\ \Sigma)$).

=10=

Ex. IF $F_2 \leq G$ THEN G HAS TARSKI NB. 4.