BANACH-TARSKI PARADOX, VON NEUMANN-DAY CONJECTURE
EX. SHEET 1 TT 2019

Answers should be sent by email to drutu@maths.ox.ac.uk, in any legible format, by 12 June at the latest.

Exercise 1. Consider an isometric copy of the unit 2-dimensional sphere $\mathbb{S}^2$ inside the unit 4-dimensional sphere $\mathbb{S}^4$ (defined for instance by identifying a point $(x, y, z) \in \mathbb{S}^2$ to the point $(x, y, z, 0, 0)$ in $\mathbb{S}^4$.

Prove that $\mathbb{S}^4 \setminus \mathbb{S}^2$ is $SO(5)$–congruent to $\mathbb{S}^4$.

Exercise 2. Let (X, dist_X) and (Y, dist_Y) be metric spaces. Prove the equivalence of the following three statements (all defining what it means for X and Y to be *quasi-isometric*):

(1) there exist two constants $L \geq 1$ and $C \geq 0$ and a map $f : X \to Y$ such that:
- $\frac{1}{L}\mathrm{dist}(x, x') - C \leq \mathrm{dist}(f(x), f(x')) \leq L\mathrm{dist}(x, x') + C$
- every point in Y is at distance at most C from a point in $f(X)$.

(2) there exist two constants $L \geq 1$ and $C \geq 0$ and two maps $f : X \to Y$ and $g : Y \to X$ such that:
- $\mathrm{dist}(f(x), f(x')) \leq L\mathrm{dist}(x, x') + C$ and $\mathrm{dist}(g(y), g(y')) \leq L\mathrm{dist}(y, y') + C$
- $\mathrm{dist}(f(g(y)), y) \leq C$ and $\mathrm{dist}(g(f(x)), x) \leq C$.

(3) there exist $N_X \subset X$ and $N_Y \subset Y$ separated nets, a constant $\lambda \geq 1$ and a map $g : N_X \to N_Y$ that is a λ–bi-Lipschitz bijection, that is, a bijection satisfying the inequalities:

$$\frac{1}{\lambda} d(x, y) \leq d(f(x), f(y)) \leq \lambda d(x, y).$$

Exercise 3. Let T_m denote the regular simplicial tree in which every vertex has valency m.

(1) Prove that for every integer $m \geq 3$, T_m is quasi-isometric to T_3. What are the explicit constants of the quasi-isometry ?
(2) Deduce that any two free groups of finite rank, F_n and F_k, endowed with any two word metrics, are bi-Lipschitz equivalent. Find explicit constants λ for the bi-Lipschitz bijection, in terms of the ranks n and k.

Exercise 4. Prove that there exists an ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ containing all the non-trivial subgroups of $\mathbb{Z}$ (*profinite ultrafilter*).

Exercise 5. Let G be a finitely generated group and S a fixed finite generating set. The goal of the exercise is to show that the Følner definition of amenability implies the existence of a mean.

Recall that the *Følner definition of amenability* requires that for every finite subset F of G and every $\epsilon > 0$ there exists Ω finite subset in G such that $\Omega \triangle F\Omega$ has cardinality at most $\epsilon\, \mathrm{card}\, \Omega$.

(1) Prove that it suffices that the Følner condition is satisfied for S and arbitrary ϵ.

(2) Let ω be an ultrafilter on $\mathbb{N}$. For every integer $n \geq 1$, let Ω_n be a set such that $\Omega_n \triangle S\Omega_n$ has cardinality at most $\frac{1}{n} \operatorname{card} \Omega_n$. Prove that the map $m : \ell^\infty \to \mathbb{R}$ defined by

$$m(f) = \lim_{\omega} \frac{1}{\operatorname{card} \Omega_n} \sum_{g \in \Omega_n} f(g)$$

is a G–left invariant mean. Above $\lim_\omega$ signifies the ω–limit of the sequence.