On graph limits and random metric spaces

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2012

Lecture 1

Several comments regarding graph metric approximations of the Euclidean metric, scaling limits of graphs and local limits of graphs.

We will discuss how well the graph metric on bounded degree graphs can approximate the metric of homogeneous manifolds equipped with some invariant length metric.

Recall that the scaling limit of the $\mathbb{Z}^2$ grid is the L^1 metric on the plane.

In the first part we will remark on graph approximations of the Euclidean (L^2) metric.

In the second part we will look at approximating invariant metrics on manifolds with a given topology.

The last part is about inducing Euclidean structure on graphs.

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The last part is about inducing Euclidean structure on graphs.

Given a graph $G = (V, E)$, the *graph distance* between any two vertices is the length of the shortest path between them.

Slack-isometry

Definition

Two metric spaces G and H are said to be *quasi-isometric* if there exists a map $f : G \to H$ and two constants $1 \leq C < \infty$ and $0 \leq c < \infty$, such that

- $C^{-1} d_H(f(x), f(y)) - c \leq d_G(x, y) \leq C d_H(f(x), f(y)) + c$ for every $x, y \in G$,
- For every $y \in H$ there is an $x \in G$ so that $d_H(f(x), y) < c$.

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Gady's question

Question

Is there a bounded degree graph which is slack-isometric to the Euclidean plane?

The Pinwheel tiling, which is a non-periodic tiling defined by Charles Radin (see Wikipedia), is a graph quasi-isometric to the Euclidean plane where the multiplicative constant goes to 1 uniformly in the distance.

The Poisson Voronoi tessellation will almost surely have an asymptotically Euclidean metric. Note that the Euclidean metric underlies the construction.

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Near critical percolation

The Gromov-Hausdorff distance between two metric spaces is obtained by taking the infimum over all the Hausdorff distances between isometric embeddings of the two spaces in a common metric space.

Can the L^2 metric "naturally" emerge as a limit of graph metrics in the Gromov-Hausdorff distance?

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Near critical percolation

Consider the natural embedding of the square grid in the plane.

Dilute the planar square grid by *removing* edges independently with probability $q < 1/2$. $1/2$ is the critical percolation probability (Kesten (80)). Thus almost surely there is a unique connected dense infinite subgrid left.

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Condition on the origin to be in the infinite connected component and look at large balls rescaled to have diameter 1.

For any fixed q the subadditive ergodic theorem was used in the context of first passage percolation to show that the rescaled large balls around the origin will a.s. converge in the *Gromov-Hausdorff* distance to a centrally symmetric convex body in the Euclidean plane.

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As $q \to 1/2$ the limiting shape Gromov-Hausdorff converges to an Euclidean ball.

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Symmetric graphs

Definition

A graph G is vertex transitive if for every two vertices in G there is an isometry of G mapping one to the other.

Symmetric graphs

Is there a sequence of finite vertex transitive graphs which Gromov-Hausdorff converges to the sphere S^2? (equipped with some invariant length metric).

Symmetric graphs

Let (G_n) be an unbounded sequence of finite, connected, vertex transitive graphs such that $|G_n| = o(\mathrm{diam}(G_n)^d)$ for some $d > 0$.

Theorem (with Hilary Finucane and Romain Tessera)

Up to taking a subsequence, and after rescaling by the diameter, the sequence (G_n) converges in the Gromov Hausdorff distance to a torus of dimension $< d$, equipped with some invariant Finsler metric.

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In particular if the sequence admits a doubling property at a small scale then the limit will be a torus equipped with some invariant length metric. Otherwise it will not converge to a finite dimensional manifold.

The proof relies on a recent quantitative version of Gromov's theorem on groups with polynomial growth obtained by Breuillard, Green and Tao and a scaling limit theorem for nilpotent groups by Pansu.

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Symmetric graphs

Establishing quantitative versions will have applications to random walks and percolation on vertex transitive graphs.

For example:

Let G be a finite, d-regular connected vertex transitive graph. View G as an electrical network in which each edge is a one Ohm conductor.

Conjecture (with Gady Kozma)

For any two vertices

$$\textit{electric resistence}(v,u) < C_d + \frac{\mathrm{diam}^2(G)\log|G|}{|G|}.$$

In addition for a sequence of vertex transitive graphs, if the diameter is $o(|G_n|)$ then the electric resistance between any two vertices is $o(\mathrm{diam}(G_n))$.

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Since finite vertex transitive graphs, when converge to a manifold, converge to tori, it follows that the infimum over all such, of the Gromov Hausdorff distance to S^n, is attained. Which one is the closest?

Is the skeleton of the *truncated icosahedron* (soccer ball) the closest to S^2?

"Proof": Otherwise we will have different design for soccer balls.

See also Géode (géométrie) in French Wikipedia.

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Roughly transitive graphs

A metric space X is (C, c)-roughly transitive if for every pair of points $x, y \in X$ there is a (C, c)-quasi-isometry sending x to y.

If G_n is only roughly transitive and $|G_n| = o(\mathrm{diam}(G_n)^{1+\delta})$ for $\delta > 0$ sufficiently small, we are able to prove, this time by elementary means, that (G_n) converges to a circle.

Question

Is there an infinite (C, c)-roughly transitive graph, with C, c finite, which is not quasi-isometric to a homogeneous space?

Where a homogeneous space is a space with a transitive isometry group.

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Going in the opposite direction

Question

Which graphs can be realized as the nerve graph of a sphere packing in Euclidean d-dimensional space?

Where vertices correspond to spheres with disjoint interiors and edges to tangent spheres.

Going in the opposite direction

The rich two dimensional theory started with Koebe, who proved that all planar graphs admits a circle packing.

In higher dimensions, Thurston observed that packability implies order $|G|^{(d-1)/d}$ upper bound on the size of minimal separators. There is an emerging theory with many still open directions.

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Metrics on graphs from packings

Three results and a question.

Theorem (with Oded Schramm)

The grid $\mathbb{Z}^4$, $T_3 \times \mathbb{Z}$ and lattices in hyperbolic 4-space cannot be sphere packed in Euclidean $\mathbb{R}^3$.

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Let (G_n) be a sequence of finite, $(k > 2)$-regular graphs with girth growing to infinity,

Theorem (with Nicolas Curien)

For every d there exists an $N(d)$ such that G_n is not regularly sphere packed in Euclidean d-dimensional space for any $n > N(d)$.

The proof uses sparse graphs limits. By regularly we mean uniform upper bound on the ratio of the radii of neighboring spheres.

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Metrics on graphs from packings

The following is an extension to higher dimension of a theorem of Bowditch (1995) for planar graphs, following a suggestion by Gromov.

Theorem (with Nicolas Curien)

Let G be an infinite locally finite connected graph which admits a regular packing in $\mathbb{R}^d$. Then we have the following alternative: either G has a positive Cheeger constant, or they are arbitrarily large subsets S of G such that $|\partial S| < |S|^{\frac{d-1}{d}+o(1)}$.

Metrics on graphs from packings

Question

Show that any packing of $\mathbb{Z}^3$ *in* $\mathbb{R}^3$ *has at most one accumulation point in* $\mathbb{R}^3 \cup \{\infty\}$.

Lecture 2

Critical and off critical metric spaces.
Some families of metric spaces are naturally parameterized by the reals. The critical spaces are usually more exotic.
We will present few examples.
These spaces sometimes admits combinations of properties which are impossible in the vertex transitive world.
We start with the classic model First Passage Percolation of for perspective.

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First Passage Percolation

Perturb the graph metric by assigning independent identically distributed length to the edges of a graph. E.g. associate to each edge an independent length having Bernoulli or exponential distribution.

Richardson (1973) proved the first shape theorem, when the length has exponential distribution and the graph is the $\mathbb{Z}^d$ lattice. The r ball around the origin, rescaled to have diameter 1, GH converges to a deterministic centredly symmetric convex shape a.s.

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The main open problems are regarding quantitative rate of convergence and geodesics in the random FPP metric on the grid.

It is conjectured that the variance of the distance from the origin to $(n, 0)$ in two dimensions is $n^{2/3}$ and it has a Tracy-Widom distribution. So far only an upperbound of $\frac{n}{\log n}$ is known. In higher dimensions it is conjectured to be much more concentrated and there are no results at all except for trees.

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First Passage Percolation

Furstenberg asked (80's) if a.s. there are two sided infinite geodesics? It is still open and even the structure of geodesic rays is only very partially understood. due to Hoffman, Damron and Newman among others.

Pertubations, beyond first passage percolation(FPP)

We now describe several random metrics, the first two can be viewed as perturbations of the grid as FPP, but with slightly stronger perturbation "causing the underling grid metric to almost disappear".

LRP

Start with the one dimensional finite grid $\mathbb{Z}/n\mathbb{Z}$ with the nearest neighbor edges, add to it additional edges as follows. Between, i and j add an edge with probability $\beta|i-j|^{-s}$, independently for any pair.

The main problem is how does the distance between 0 and n grows typically in this random graph?

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LRP

The off critical cases:
When $s > 2$ distance is order n, when $1 < s < 2$ distance is polylog$n$ (Biskup (2003)), when $s = 1$ distance is $\frac{\log n}{\log \log n}$ (Coppersmith, Gamarnik and Sviridenko (2002)) . and when $s < 1$ distance is uniformly bounded(Benjamini, Kesten, Peres and Schramm (2004)).

LRP

The critical case:

When $s = 2$ the distance is of the form $\theta(n^{f(\beta)})$, where f is strictly between 0 and 1 (Sly and Ding (2012)).

Continuity, monotonicity, or even a guess for f are still open.

These natural random graphs looks very different from vertex transitive graphs. E.g. when $1 < s < 2$ the mixing time of simple random walk is a.s. n^{s-1}.

That is, small diameter do not exclude small bottlenecks as in vertex transitive graphs.

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CCCP

Examine bond percolation on $\mathbb{Z}^d$. Each edge is open with probability p independently. Clusters are connected components of open edges. For any $d > 1$, there is $0 < p_c < 1$, such that if $p < p_c$ all the clusters are finite a.s. and the diameter of the clusters has exponential tail. If $p > p_c$ there is a unique infinite cluster. While for the critical probability p_c it is conjectured that there is no infinite cluster and diameter of clusters has polynomial tail. True in dimensions 2 and d large.

CCCP

The unique infinite cluster, for $p > p_c$ is a random perturbation of the grid. E.g. asymptotics of the heat kernel are the same, how can we get "interesting" critical geometry?

Conditioning on the critical percolation to have an infinite cluster results in a graph with infinitely many cut points.

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Conditioning on the critical percolation to have an infinite cluster results in a graph with infinitely many cut points.

Here is a suggestion:

Contract each cluster into a single vertex. The result is a random graph G with high degrees (each vertex $v \in G$ is a cluster $\mathcal{C}$ in $\mathbb{Z}^d$ and its degree is the number of closed edges coming out of $\mathcal{C}$). When the percolation is *subcritical* one expects to see a perturbation of the lattice. A random version of a rough isometry, but when the percolation is *critical* the random geometric structure obtained is expected to be rather different.

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We say CCCP instead of G (CCCP standing for Contracting Clusters of Critical Percolation).
For example (with Ori Gurel-Gurevich and Gady Kozma) we have:
When $d = 2$, the CCCP has exponential volume growth a.s.
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Random subdivisions

There is growing interest in establishing a rigorous theory of two dimensional continuum quantum gravity. Heuristically, quantum gravity is a metric chosen on the sphere uniformly among all possible metrics. Although there are successful discrete mathematical quantum gravity models, we do not yet have a satisfactory continuum definition of a planar random length metric space (rather than random measure).

Random subdivisions

One possible toy model is to start with a unit square divide it four squares and now recursively at each stage pick a square uniformly at random from the current squares (ignoring their sizes) and divide it to four squares and so on.

Random subdivisions

Look at the minimal number of squares needed in order to connect the bottom left and top right corner with a connected set of squares.

We *conjecture* that there is a deterministic scaling function, such that after dividing the random minimal number of squares needed after n subdivisions by it, the result is a non degenerate random variable. Establishing the conjecture will provide a random planar length metric space.

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Stationary graphs

A distribution on rooted graphs is called *stationary* if it invariant under re-rooting a long a simple random walk path.
This is weakening of the notion of vertex transitive graphs.
Examples include limits of finite graphs with the root chosen proportional to the degree. In particular graphs obtained using a subdivision rule.
Try to imagine planar triangulation with polynomial volume growth of balls around the root which is faster than quadratic. Try to imagine such a growth around each vertex of the triangulation or stationary planar triangulation's with polynomial volume growth which is faster than quadratic.

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Cuts in planar graphs

Theorem (with Panos Papasoglu)

Let G be a planar graph such that the volume function of G satisfies $V(2n) < CV(n)$ for some constant $C > 0$. Then for every vertex v of G and integer n, there is a domain Ω such that $B(v,n) \subset \Omega$, $\Omega \subset B(v,6n)$ and the size of the boundary of Ω is at most order n.

That is, for a volume doubling planar graphs even with polynomial growth faster than quadratic there are still linear cuts, unlike say the 3 dimensional grid.

How can this be combined with stationarity and large growth?

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This suggests that such a graph has a fractal structure of cactus like folds at all scales in order to account for the volume together with the small cuts as seen from every point.
The facts above suggest heuristically that volume is generated by large fractal mushrooms, and that the complements of balls have many connected components.

Cuts in planar graphs

We *conjecture* that the simple random walk spend a long time is such traps and hence is subdiffusive (that is, $\mathrm{dist}(o, X_n) \asymp n^{\alpha}$ where $X(n)$ denotes the simple random walk starting at o and $\alpha < 1/2$) ?

Note that no such small cuts in the context of vertex transitive graphs.
E.g. Aldous proved that for any finite vertex transitive graph the isoperimetric constant is at least order $\frac{1}{\text{diameter}}$.

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Limits

A limit of finite graphs G_n is a random rooted infinite graph (G, ρ) with the property that neighborhoods of G_n around a random vertex converge in distribution to neighborhoods of G around ρ.

Formally, let (G, o) and $(G_1, o_1), (G_2, o_2), \ldots$ be random connected rooted locally finite graphs. We say that (G, o) is the limit of (G_j, o_j) as $j \to \infty$ if for every $r > 0$ and for every finite rooted graph (H, o'), the probability that (H, o') is isomorphic to a ball of radius r in G_j centered at o_j converges to the probability that (H, o') is isomorphic to a ball of radius r in G centered at o.

A random rooted finite graph (G, o) is unbiased, if given G the root o is uniformly distributed among the vertices of G. In particular given a (possibly random) graph we will consider the distribution on rooted graphs obtained by rooting at a random uniform vertex.

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Exercise: What is the limit of n-levels full binary trees?

Hint: A random uniform vertex in the n-levels full binary, is near the leaves.

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The limit of the n-levels full binary tree is the *canopy* tree, consisting of half line $\mathbb{N}$ with a binary tree of height n rooted at n and the root is on the line with geometric$(1/2)$ distance to 0.

This example illustrates the following, With Oded Schramm we proved(2001) that limits of bounded degree finite planar graphs are a.s. recurrent for the simple random walk.

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Limits of triangulations with no genus restriction

Guth, Parlier and Young (2010) studied pants decomposition of random closed surfaces obtained by randomly gluing N Euclidean triangles (with unit side length) together.

They gave bounds on the size of pants decomposition of random compact surfaces with no genus restriction as a function of N. Their work indicates that the injectivity radius around a typical point is growing to infinity.

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Limits of triangulations with no genus restriction

Take a uniform measure on triangulations with N triangles conditioned on the genus to be CN for some fixed $C < 1/4$ and a uniformly chosen root.

We *conjecture* that as N grows, this random surface converges to a rooted random triangulation of the hyperbolic plane with average degree $6/(1-4C)$.

In particular we believe that the local injectivity radius around typical vertex will go to infinity on such a surface as N grows.

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Stochastic Hyperbolic Infinite Quadrangulation

The *Uniform infinite planar quadrangulation* (UIPQ) (Angel and Schramm, Krikun) is analogous to the limit of random planar triangulation (zero genus) for quadrangulation.

Philippe Chassaing constructed the UIPQ via Schaeffer's bijection from a labeled *critical* Galton-Watson tree conditioned to survive.

We propose the study of an infinite random quadrangulation constructed similarly from a labeled *super* critical Galton-Watson trees.

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Shiq

We know that simple random walk on the Shiq has positive speed a. s.

Unlike the zero genus UIPQ which is recurrent (Gurel-Gurevich and Nachmias) and sub diffusive (with Curien), basic properties of the Shiq are still unknown.

Hyperfinite

Gábor Elek introduced the notion of a *hyperfinite* graph family: a collection of graphs is hypefinite if for every $\epsilon > 0$ there is some finite k such that each graph G in the collection can be broken into connected components of size at most k each has boundary of size at most ϵ of it's size.

Theorem (Oded Schramm)

If a sequence of finite graphs converges to a hyperfinite limit, then the sequence itself is hyperfinite.

The Shiq is not hyperfinite.

Lecture 3

More on geometry and probability.

Connective constant

Connective constants for self avoiding walks admit some partial analogy with the critical probability of percolation. Both monotone with respect to inclusion and graph covering. With Hugo Duminil-Copin we briefly formulate the analogous conjectures in the context of self avoiding walks:

Let G be a graph (for concreteness one can think on $\mathbb{Z}^d$). Self avoiding walk (SAW) is a random walk that does not return to a vertex that he already visits.

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Connective constant

Define $SAW(n)$ as the uniform measure on all the avoiding paths walk of length n from a fixed root. By sub multiplicativity $\mu = \lim |SAW(n)|^{1/n}$ exists and is called the *connective constant* of the graph.

Connective constant

Conjecture

There is $c > 1$, $\mu > c$ for all infinite connected vertex transitive graph excluding $\mathbb{Z}$.

Maybe the ladder is a graph with the smallest connective constant other than $\mathbb{Z}$, among all vertex transitive? Otherwise consider an infinite path and add a parallel edge to every second edge, to get $\mu = \sqrt{2}$.

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Connective constant

One way to establish the conjecture is to show that every infinite vertex transitive graph covers an infinite vertex transitive graph of girth bounded by a fixed constant. As for all such graph it easy to show a uniform lower bound, yet large girth seems only to help. This is likely not the case but hopefully leaves a small family of graphs to be studied. (Yair Glasner suggested that, due to Margulis super rigidity, maybe Cayley graphs of $SL_3(\mathbb{Z})$ do not cover other infinite graphs?)

Connective constant

A stronger conjecture is,

Conjecture

μ is continuous with respect to local convergence of infinite vertex transitive graphs.

Given a Cayley graph, for any generating set corresponds a connective constant μ. This suggests a canonical generating sets minimizing μ. Gady Kozma conjectured that for planar Cayley graphs μ is algebraic, and he showed that the set of all connective constants of groups contains an interval.

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Scale invariance

Expander at all scales?

Definition

Let $G = (V, E)$ be a finite graph. Define the Cheeger constant of G to be

$$h(G) = \inf_{0<|S|<\frac{|V|}{2}} \frac{|\partial S|}{|S|}.$$

If G is an infinite graph we set

$$h(G) = \inf_{0<|S|<\infty} \frac{|\partial S|}{|S|}.$$

An infinite graph G with $h(G) > 0$ is called non-amenable. Otherwise it is called amenable.

A sequence of graphs $\{G_n\}$ is of a uniform expander if there is $h > 0$, for all n, $h(G_n) > h$.

Expander at all scales?

Question

Is there a family $\{G_n\}$ of finite d-regular graphs, $|G_n| \to \infty$, so that all balls in all the G_n's are uniform expanders?

That is, there is $h > 0$, for all $r > 0$ and any v in any of the graphs G_n's the ball $B(v, r)$ is h- expander, expander with a uniform edge expansion constant h. Note e.g. that if G_n is a sequence of expanders with girth growing to infinity, then if r is smaller than the girth then the balls of radius r are trees and thus not uniform expanders as r grows.

We *conjecture* that there is no such family. If there is it will give a "useful" new network architecture.

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Expander balls

For vertex transitive graphs a positive answer to the following conjecture regarding percolation on expanders will show that no such graph exists. The proof will proceed by constructing a limiting nonamenable vertex transitive graph with a unique infinite cluster whenever percolation occurs.

Question

Let G be a bounded degree expander, further assume that there is a fixed vertex $v \in G$, so that after performing $p = 1/2$ percolation on G,

$$P_{1/2}(\mathit{diam(connected\ component\ of\ v)} > \mathit{diam}(G)/2) > 1/2,$$

Show that there is a giant component w.h.p? G is not assumed to be transitive.

Expander balls

The following two questions are regarding the rigidity of the global structure given local information.

Question

Given a fixed rooted ball $B(o, r)$, assume there is a finite graph such that all its r-balls are isomorphic to $B(o, r)$, e.g. $B(o, r)$ is a ball in a finite vertex transitive graph, what is the minimal diameter of a graph with all its r-balls isomorphic to $B(o, r)$? Any bounds on this minimal diameter, assuming the degree of o is d? Any example where it grows faster than linear in r, when d is fixed?

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Expander balls

Few comments,
Note that a 3-ball in the grandparent graph does not appear as a ball in a finite vertex transitive graph.

When the rooted ball is a tree, this is the girth problem.

One can consider a weaker version e.g. when we require only that most balls are isomorphic to $B(o, r)$.

Not assuming a bound on the degree, consider the 3-ball in the hypercube, is there a graph with a smaller diameter than the hypercube so that all its 3-balls are that of the hypercube?

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Expander balls

Question (with Romain Tessera)

Let X is the Euclidean or the hyperbolic plane, together with a triangulation, whose triangles are at most of diameter r. Suppose for each pair of Euclidean (or hyperbolic) balls of radius r, B_1, B_2 centered on vertices of this triangulation, there is a Euclidean (or hyperbolic) isometry mapping B_1 to B_2 respecting the triangulation (in the obvious way).

Does it imply that the triangulation is periodic?

Percolation on groups

Dilute a graph by keeping each edge independently with probability p. Look at the events: "there are infinite connected components" and "there is a unique infinite component".

Denote the critical probability for the first event p_c and for the second p_u.

For vertex transitive graphs uniqueness monotonicity holds.

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For which graphs there is a critical probability strictly between 0 and 1?

For which graphs $p_c < p_u$? For which graphs $p_u < 1$?

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It is *conjectured* that for nonamenable vertex transitive graphs $p_c < p_u$.

Pak and Nagnibeda proved that for any nonamenable Cayley graph there are generating sets for which $p_c < p_u$.
This is relevant to question if there is a Cayley graph with expander balls.

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A long standing problem it is quasi isometric invariant in the context of vertex transitive graphs?

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Random foliations

Can you foliate $\mathbb{R}^d$ with Brownian paths?