Embeddings and higher index theory

As notes, I recommend the book I wrote with my former student Piotr Nowak on the subject.

1 Background

1.1 Topological rigidity

In the area of recognition and classification of manifolds, rigidity means the following.

Definition 1 *A compact manifold M is rigid if any N which is homotopy equivalent to M must be homeomorphic to M.*

Example 1 1 *and* 2*-manifolds are rigid.* S^3 *is rigid (Perelman). Lens spaces are not rigid.*

Lens spaces are quotients of the 3-sphere by cyclic groups of isometries. Given two relatively prime integers p and q, let

$$h(z_1, z_2) = (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2).$$

Then the cyclic group generated by h acts freely on the unit sphere. Let $L(p, q)$ be the quotient space. It is a manifold.

One shows that $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent. To show that they are not homeomorphic, one uses Rademeister torsion.

1.2 Asphericity

To put such examples out of our way, we shall restrict to *aspherical manifolds*.

Definition 2 *A manifold M is aspherical if its universal cover is contractible.*

Example 2 *Tori are aspherical.*

This helps us, but does not make life that much easier. For instance, rigidity of T^3 implies the Poincaré conjecture.

Conjecture (Borel): All compact aspherical manifolds are rigid.

Note that for aspherical manifold, the homotopy type is determined by the fundamental group. So Borel conjecture states that the topological type would be entirely encoded in the the fundamental group. Sounds hard.

1.3 Cayley graphs

Our brain is designed to recognize things from pictures. Our picture of a fundamental group is a Cayley graph.

Definition 3 *Let G be a group with finite generating set S. The Cayley graph is the graph with vertex set G, where an edge connects g_1 and g_2 iff $g_1^{-1}g_2 \in S$.*

Example 3 $\mathbb{Z} \to$ *line, free group F_2 on two generators $\to$ tree.*

1.4 Coarse embeddings

A Cayley graph comes with a metric, which usually does not embed nicely in the plane or finite dimensional Euclidean space: F_2 often complains about his common pictures as a tree with shorter and shorter edges. On the other hand, the Cayley graph of F_2 has a natural embedding in infinite dimensional Hilbert space, where every edge is parallel to a different direction. It is not isometric, but not that bad: For x, $y \in \Gamma$, Hilbert and graph metrics are related by

$$d_H(x,y) = \sqrt{d(x,y)}.$$

Definition 4 (Gromov) *A map $f : G \to H$ is a coarse embedding if there exist two non-decreasing functions ρ_1 and ρ_2 on $[0,\infty)$ such that*

1. $\forall x,\, y \in G$,

$$\rho_1(x,y) \leq d_H(f(x), f(y)) \leq$$

2.

$$\lim_{r\to\infty} \rho_i(r) = +\infty.$$

The goal of these lectures is to explain how coarse embeddability of the fundamental group in Hilbert space implies variations of Borel's conjecture.

1.5 Stable Borel conjecture

Definition 5 *A compact manifold M is stably rigid if whenever a compact manifold N is homotopy equivalent to N, $N \times \mathbb{R}^n$ is homeomorphic to $M \times \mathbb{R}^n$ for some n.*

Conjecture: All compact aspherical manifolds are stably rigid.

This is still open, but a closely related result exists.

Theorem 6 (Guentner-Tessera-Yu) *Let M be a compact aspherical manifold. If the fundamental group G has constructive coarse embeddability into Hilbert space, then M is stably rigid.*

Tomorrow, I will explain what constructive coarse embeddability means and give examples.

1.6 Novikov's conjecture

Next we proceed to an infinitesimal version of the stable Borel conjecture. This requires smoothness, and a tangent bundle. We concentrate on invariants of the tangent bundle, its Pontryagin classes.

Theorem 7 (Novikov) *The rational Pontryagin classes of the tangent bundle are homeomorphism invariants of compact smooth manifolds.*

This is a hard theorem: no easy proof, even now.

Borel conjecture and Novikov's theorem would imply that rational Pontryagin classes of compact aspherical classes are homotopy invariants. This statement is known as **Novikov's conjecture** (technically speaking: in the special case of aspherical manifolds; the general case involves higher signatures, which boil down to Pontryagin classes in the aspherical case).

Theorem 8 *Let M be a compact manifold. Assume that the fundamental group is coarsely embeddable into Hilbert space. Then the Novikov conjecture holds.*

Our theorem is more general: Hilbert space can be replaced with more general Banach spaces (but not all).