Today I explain what constructive coarse embeddability means. I start with a more intuitive concept, asymptotic dimension.

1 Asymptotic dimension

2 Definition

Definition 1 (Gromov) *The asymptotic dimension of metric space X is the smallest integer n such that for each $r > 0$, there exists a uniformly bounded cover $\{U_i\}_i$ of X, i.e.*

1. *the diameters of U_i's are bounded,*
2. *each r-ball intersects at most $n+1$ members of the covering.*

Example 1 *$\mathbb{Z}$ has asymptotic dimension 1, $\mathbb{Z}^d$ has asymptotic dimension d. A regular tree has asymptotic dimension 1.*

Proposition 2 *Hyperbolic groups have finite asymptotic dimension.*

Linial: A similar concept arises in theoretical computer science.

2.1 Alternative definition

Proposition 3 *Asymptotic dimension of X is the smallest integer n such that for all $r > 0$, X can be covered by $n+1$ subset X_i, each of which being a disjoint union of r-separated subsets X_{ij}, of bounded diameters.*

This definition is very flexible. I will give examples of asymptotic dimension estimates.

2.2 $Sl(n, \mathbb{Z})$

Let us start with the group $G = Sl(n, \mathbb{Z})$. It is a subgroup of the continuous group $Gl(n, \mathbb{R})$. Equip $Gl(n, \mathbb{R})$ with a left-invariant Riemannian or Finsler metric (up to quasi-isometry, the choice of such a metric is irrelevant).

Use Gram-Schmidt orthogonalization to write any matrix $S \in Gl(n, \mathbb{R})$ as a product

$$S = TU,$$

where T is lower triangular and U is orthogonal. The map $S \mapsto T$ is a quasi-isometry of $Gl(n, \mathbb{R})$ to the group $T(n, \mathbb{R})$ of upper triangular matrices. This leads to

$$asdim(Gl(n, \mathbb{R})) = \operatorname{asdim}(T(n, \mathbb{R})).$$

$T(n, \mathbb{R})$ is a semi-direct product of an abelian group (diagonal matrices) with a nilpotent group (unipotent matrices). This can be used to show that $\operatorname{asdim}(T(n, \mathbb{R})) < \infty$. This implies that $\operatorname{asdim}(Sl(n, \mathbb{Z})) < \infty$.

The exact value of $\operatorname{asdim}(Sl(n, \mathbb{Z}))$ is not exactly known. It is conjectured to equal the cohomological dimension.

Question. Let M be a simply connected nonpositively curved manifold. Is $\operatorname{asdim}(M) < \infty$?

There are results for subclasses (e.g. symmetric spaces, strictly negative curvature), and also for classes of CAT(0) spaces: buildings, cube complexes.

2.3 A weird example

Let π be a transcendental number. Let $\Gamma \subset T(2, \mathbb{R})$ be the subgroup of matrices of the form

$$\begin{pmatrix} \pi^k & \sum_{\ell=-m}^{m} n_\ell \pi^\ell \\ 0 & \pi^{-k} \end{pmatrix}, \quad k \in \mathbb{Z}, \quad n_\ell \in \mathbb{Z}.$$

Γ is finitely generated, but not discrete. Equip it with the word metric. Let Γ_0 be the subgroup where $k = 0$. It is free abelian of infinite rank. The metric induced on Γ_0 is an ℓ^1 metric. Γ_0 contains a quasi-isometric $\mathbb{Z}^d$ for every d, so $\operatorname{asdim}(\Gamma_0) = \infty$ and $\operatorname{asdim}(\Gamma) = \infty$ as well.

Γ is also known as the wreath product $\mathbb{Z} \wr \mathbb{Z}$.