1 Proof of stable rigidity

The rigidity problems reduces to some algebra: First from homotopy to simple homotopy (collapsing cells) involves algebraic K-theory. Then from simple homotopy to homeomorphisms involves different algebra. Both steps are achievable by splitting space in controlled pieces.

The vocabulary is complicated, but the mechanism is elementary: Mayer-Vietoris allows to exploit a decomposition of space.

2 Proof of Novikov conjecture

For this infinitesimal version, one can improve the result. The invariant can be encoded in a differential equation. I will explain how differential equations can help in topology. Speculation: possible connection with high dimensional expanders.

2.1 Dirac operator

It is a differential operator D whose square equals Laplacian. There is no such scalar operator, but if one allows vector valued operators (i.e. matrices), then a solution exists. The key to the solution on $\mathbb{R}^n$ is a set of n matrices c_i such that $c_i^2 = -1$ and $c_i c_j + c_j c_i = 0$. Then set $D = \sum_i c_i \frac{\partial}{\partial x_i}$.

It turns out that D can be globally defined on manifolds (under a mild topological assumption, similar to orientability, called spin), see the nice book by H.B. Lawson and M. Michelson. Then D^2 is not exactly equal to the Levi-Civita connection Laplacian, there is an extra curvature term, due to non commutation of covariant derivatives,

$$D^2 = \text{Laplacian } + \frac{1}{4} \text{ scalar curvature.}$$

If scalar curvature is positive, this shows that D is invertible. This motivates us to investigate when D is invertible. This is a rather untractable question, answer changes when the metric is deformed. Something does not change, it is the Fredholm index of D.

2.2 Index theory

By definition

$$\text{Index}(D) = \dim(\ker(D)) + \dim(\text{coker}(D)).$$

M. Atiyah and I. Singer's Index Theorem is a formula that expresses $\mathrm{Index}(D)$ in terms of a characteristic number (a sophisticated version of Euler number, see J. Milnor's book on Characteristic classes) denoted by $\hat{A}(M)$,

$$\mathrm{Index}(D) = \hat{A}(M).$$

This is a deep theorem, there are now elegant proofs, but they require lots of background.

Example 1 *For the torus T^n, $\hat{A} = 0$. Indeed, for flat metrics,* $\ker(D)$ *and* $\mathrm{coker}(D)$ *can be determined explicitely.*

Note that the Index Theorem alone is not sufficient to show that T^n has no metric with positive scalar curvature. One needs use the interaction with the fundamental group, and work on the universal cover, this is work by M. Gromov and H.B. Lawson in the early 80's.

2.3 Higher index theory

Lift metric and Dirac operator to the universal cover $\tilde{M}$. Positivity of scalar curvature implies that $\tilde{D}$ is invertible. The fundamental group G acts on $\tilde{M}$, it commutes with $\tilde{D}$ so it acts on $\ker(\tilde{D})$, this yields a linear representation of G. This is a finer information than the mere dimension of $\ker(\tilde{D})$ (which is usually infinite...). Due to infinite dimensionality, $\ker(\tilde{D})$ must be replaced by something that takes into account the spectrum near zero. Therefore the Grothendieck ring $R(G)$ of representations of G must be replaced by some K-theory group $K_*(C_r^*(G))$. This is what the term higher index theory refers to.

2.4 Geometric Novikov conjecture

The strong Novikov conjecture requires an algorithm for deciding when $\mathrm{index}(\tilde{D})$ is non zero. It belongs to group theory.

One can also formulate an algorithm for arbitrary non compact manifolds, without group actions. Here is an unusual non compact manifold: the disjoint union of all round spheres S^k with radius k. The scaling is in order that scalar curvature is bounded from below. So D is invertible, there is even a spectral gap. It is a counterexample to the conjecture, since the topological index (element in a K-homology group) does not vanish. I view this example as a higher dimensional expander.

Question. Construct a similar counterexample with bounded dimension.

3 Embeddability in Banach spaces

This is joint work with G. Kasparov.

Definition 1 *Let X be a Banach space. Say X has property (H) if there exists a sequence of finite dimensional subspaces $V_n \subset X$ (resp. $W_n \subset H$ Hilbert space), whose union V is dense in X (resp. H), and there exists a uniformly continuous map from the unit sphere $S(V)$ to $S(W)$, $W = \bigcup_n W_n$ which is a homeomorphism of $S(V_n)$ onto $S(W_n)$ for all n.*

Example 2 *$X = \ell^p$, $H = \ell^2$, with obvious V_n and W_n, and ϕ is the Mazur map*

$$\phi(c_0, c_1, \ldots) = (\mathrm{sgn}(c_0)|c_0|^{p/2}, \mathrm{sgn}(c_1)|c_1|^{p/2}, \ldots).$$

Open question. Let c_0 be the space of sequences which tend to 0. Does c_0 have property (H) ?

If it were true,

Theorem 2 *If G admits a coarse embedding into a Banach space X with property (H), then the strong Novikov conjecture holds for G.*

Expanders do not coarsely embed in Hilbert space. Nevertheless, we can handle Novikov conjecture

Bourdon: Are there groups which coarsely embed in spaces with property (H). I answer by other uestions. B. Johnson and his student Lava ? show that ℓ^p does not coarsely embed in ℓ^2 if $p > 2$. Mendel and Naor show that ℓ^p does not coarsely embed in ℓ^q if $p > q \geq 2$.

Open question. If $p > q \geq 2$, find a bounded degree graph which coarsely embeds in ℓ^q but not in ℓ^p. Find a group with the same property.

Example 3 *Let*

$$C_p = \{T : H \to H\,;\, \mathrm{trace}((T^*T)^{p/2}) < \infty.$$

Then C_p has property (H).

I believe that our result covers groups occurring naturally in nature. For instance, let N be a compact smooth manifold and G a finitely generated group of diffeomorphisms of N.

Conjecture. G is coarsely embeddable into C_p for some p.