Solution (#32) Note

$$\begin{aligned} |1+i| &= \sqrt{2} \qquad \text{and} \qquad \arg(1+i) = \pi/4; \\ \left|\sqrt{3}+i\right| &= 2 \qquad \text{and} \qquad \arg\left(\sqrt{3}+i\right) = \pi/6. \end{aligned}$$

Hence

$$\left|\frac{1+i}{\sqrt{3}+i}\right| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \arg\left(\frac{1+i}{\sqrt{3}+i}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

Further

$$\frac{1+i}{\sqrt{3}+i} = \left(\frac{1+i}{\sqrt{3}+i}\right)\left(\frac{\sqrt{3}-i}{\sqrt{3}-i}\right) = \frac{(1+\sqrt{3})+i\,(\sqrt{3}-1)}{4}.$$

Hence

$$\frac{1}{\sqrt{2}}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{4} \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{4},$$

to give the required expressions. As an alternative approach we also have that

$$\begin{aligned} 2\cos^2\frac{\pi}{12} - 1 &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \implies &\cos\frac{\pi}{12} = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}} = \frac{1}{2}\sqrt{2+\sqrt{3}}, \end{aligned}$$

taking the positive square root as $\cos(\pi/12) > 0$. As $\left(2+\sqrt{3}\right) = \frac{1}{2}\left(1+\sqrt{3}\right)^2$ the two formulae for $\cos(\pi/12)$ are in agreement.