Solution (#40) By the fundamental theorem of algebra

$$P(z) = A(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_k).$$

for some non-zero complex number A. Hence, by the product rule of differentiation,

$$P'(z) = A(z-\alpha_2)\cdots(z-\alpha_k) + \cdots + A(z-\alpha_1)\cdots(z-\alpha_{k-1})$$

and

$$\begin{aligned}\frac{P'(z)}{P(z)} &= \frac{A(z-\alpha_2)\cdots(z-\alpha_k) + \cdots + A(z-\alpha_1)\cdots(z-\alpha_{k-1})}{A(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_k)} \\ &= \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + \cdots + \frac{1}{z-\alpha_k}\end{aligned}$$

Suppose now that $\operatorname{Im}\alpha_i > 0$ for each i and for a contradiction that $\operatorname{Im}\beta \leqslant 0$. Then $\operatorname{Im}(\beta-\alpha_i) < 0$ and hence

$$\operatorname{Im}\left(\frac{1}{\beta-\alpha_i}\right) > 0.$$

Then

$$\operatorname{Im}\frac{P'(\beta)}{P(\beta)} = \operatorname{Im}\left(\frac{1}{\beta-\alpha_1}\right) + \operatorname{Im}\left(\frac{1}{\beta-\alpha_2}\right) + \cdots + \operatorname{Im}\left(\frac{1}{\beta-\alpha_k}\right) > 0$$

and in particular $P'(\beta) \neq 0$. This is a contradiction and hence $\operatorname{Im}\beta > 0$.

More generally, if all the roots of a polynomial lie in half-plane $\operatorname{Im}((z-a)/b) > 0$ (see #38) then we can make a change of variable $w = (z-a)/b$. In terms of the new co-ordinate w the plane is now the region $\operatorname{Im} w > 0$ whilst the polynomial $P(z)$ equals

$$P(bw + a)$$

which is remains a polynomial (in w) and we can apply the previous part.