Solution (#57) The modulus of -1 is 1 and its argument is π. Hence its seventh roots are

$$z_k = \operatorname{cis} \frac{(2k+1)\,\pi}{7} \qquad (0 \leqslant k \leqslant 6)\,.$$

On factorizing $z^7 + 1$ we have

$$z^7 + 1 = (z - z_0)\,(z - z_1) \cdots (z - z_6)\,.$$

If we compare the coefficients of z^6 in both sides we see

$$0 = -\left(z_0 + z_1 + \cdots + z_6\right).$$

By taking real parts we have

$$\begin{aligned} 0 &= \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} + \cos\pi + \cos\frac{9\pi}{7} + \cos\frac{11\pi}{7} + \cos\frac{13\pi}{7} \\ &= 2\left(\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}\right) - 1 \end{aligned}$$

as $\cos(\pi/7) = \cos(13\pi/7)$, etc. Rearranging this we get the required result.