Solution (#61) If we define the polynomial

$$f(x) = 64x^7 - 112x^5 + 56x^3 - 7x + 1$$

then we see that

$$f(\cos(\pi/7)) = \cos 7(\pi/7) + 1 = \cos \pi + 1 = 0$$

so that $\cos(\pi/7)$ is a root of $f(x)$. Similarly $\cos(3\pi/7)$, $\cos(5\pi/7)$, $\cos \pi = -1$ are all roots of $f(x)$. Further

$$\begin{aligned} \frac{f(x)}{x+1} &= 64x^6 - 64x^5 - 48x^4 + 48x^3 + 8x^2 - 8x + 1 \\ &= \left(8x^3 - 4x^2 - 4x + 1\right)^2 \end{aligned}$$

so that the roots $\cos(\pi/7), \cos(3\pi/7)$, $\cos(5\pi/7)$, are all repeated roots of x.

If we denote the seven roots of $f(x)$ as $\alpha_1, \ldots, \alpha_7$ then

$$f(x) = 64(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_7)$$

and so comparing the constant coefficients

$$-64\alpha_1\alpha_2 \cdots \alpha_7 = 1.$$

As we know these roots then

$$-64 \cos^2(\pi/7) \cos^2(3\pi/7) \cos^2(5\pi/7) \cos \pi = 1.$$

Hence

$$\cos(\pi/7) \cos(3\pi/7) \cos(5\pi/7) = \pm 1/8.$$

As $\cos(\pi/7) > 0, , \cos(3\pi/7) > 0$, $\cos(5\pi/7) < 0$, then

$$\cos(\pi/7) \cos(3\pi/7) \cos(5\pi/7) = -1/8.$$