Solution (#71) (i) and (ii) We have $z \neq 0$ and $q = m/n$ be rational. So we may write $z = r \operatorname{cis}\theta$ for some choice of argument θ. Let

$$w = r^{m/n} \operatorname{cis}\left(\frac{m\theta}{n}\right).$$

Note that $w^n = z^m$ by De Moivre's theorem. Further if α is any of the n roots of unity we have $(\alpha w)^n = \alpha^n w^n = z^m$ as well.

On the other hand if we also have $\zeta^n = z^m$ then $(\zeta/w)^n = 1$ and so $\zeta = \alpha w$ where α is an nth root of unity and so we've shown

$$z^q = \left\{Z, \omega Z, \omega^2 Z, \ldots, \omega^{n-1} Z\right\}$$

where $\omega = \operatorname{cis}(2\pi/n)$ as $\left\{1, \omega, \ldots, \omega^{n-1}\right\}$ is the set of nth roots of unity.

(iii) Let k be an integer. By Definition 1.24,

$$z^k = z^{k/1} = \left\{w \in z \colon w^1 = z^k\right\} = \left\{z^k\right\}.$$