Solution (#75) (i) Let $z = \operatorname{cis}\theta$ and n be an integer. Then

$$\frac{1}{z} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$$

and hence $z + z^{-1} = 2\cos\theta$ and $z - z^{-1} = 2i\sin\theta$.

(ii) So by De Moivre's theorem

$$\begin{aligned} 2\cos n\theta &= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = z^n + z^{-n}, \\ 2i\sin n\theta &= (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = z - z^{-n}. \end{aligned}$$

(iii) Hence

$$\begin{aligned} \cos^5\theta &= \frac{1}{32}\left(z + z^{-1}\right)^5 \\ &= \frac{1}{32}\left(z^5 + z^{-5} + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right)\right) \qquad \text{[by the binomial theorem]} \\ &= \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right). \end{aligned}$$

(iv) Finally

$$\begin{aligned} \int_0^{\pi/2} \cos^5\theta \, \mathrm{d}\theta &= \frac{1}{16}\int_0^{\pi/2} \left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right) \mathrm{d}\theta \\ &= \frac{1}{16}\left[\frac{\sin 5\theta}{5} + \frac{5\sin 3\theta}{3} + 10\sin\theta\right]_0^{\pi/2} \\ &= \frac{1}{16}\left(\frac{1}{5} - \frac{5}{3} + 10\right) \\ &= \frac{8}{15}. \end{aligned}$$