Solution (#77) By De Moivre's theorem, for any real value x and natural number $n \geqslant 0$,

$$\begin{aligned}
& \sum_{k=0}^{n} \cos kx + i \sin kx \\
= & \sum_{k=0}^{n} (\cos x + i \sin x)^k \\
= & \frac{1 - (\cos x + i \sin x)^{n+1}}{1 - \cos x - i \sin x} \\
= & \frac{1 - \cos(n+1)x - i \sin(n+1)x}{1 - \cos x - i \sin x} \\
= & \frac{2\sin^2 \frac{(n+1)x}{2} - 2i \sin \frac{(n+1)x}{2} \cos \frac{(n+1)x}{2}}{2\sin^2 \frac{x}{2} - 2i \sin \frac{x}{2} \cos \frac{x}{2}} \qquad \text{[using double angle formulae]} \\
= & \frac{\sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}} \left\{ \frac{\sin \frac{(n+1)x}{2} - i \cos \frac{(n+1)x}{2}}{\sin \frac{x}{2} - i \cos \frac{x}{2}} \right\} \\
= & \frac{\sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}} \left\{ \left(\sin \frac{(n+1)x}{2} - i \cos \frac{(n+1)x}{2} \right) \left(\sin \frac{x}{2} + i \cos \frac{x}{2} \right) \right\} \\
= & \frac{\sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}} \left\{ \cos \frac{nx}{2} + i \sin \frac{nx}{2} \right\} \qquad \text{[using the } \cos(A+B) \text{ and } \sin(A+B) \text{ formulae].}
\end{aligned}$$

Taking real and imaginary parts gives the required formula and also shows

$$\sum_{k=0}^{n} \sin kx = \frac{\sin \frac{(n+1)x}{2} \sin \frac{nx}{2}}{\sin \frac{x}{2}}.$$