Solution (#78) We have from #77 that

$$\sum_{k=0}^{n} \cos kx = \left(\sin \frac{(n+1)\,x}{2} \cos \frac{nx}{2}\right)\left(\sin \frac{x}{2}\right)^{-1}$$

Differentiating with respect to x by the product and quotient rules, the RHS becomes

$$\begin{aligned} & \frac{\sin \frac{x}{2}\left(\frac{(n+1)}{2} \cos \frac{(n+1)x}{2} \cos \frac{nx}{2} - \frac{n}{2} \sin \frac{(n+1)x}{2} \sin \frac{nx}{2}\right) - \frac{1}{2} \cos \frac{x}{2} \sin \frac{(n+1)x}{2} \cos \frac{nx}{2}}{\sin^2 \frac{x}{2}} \\ = & \frac{n \sin \frac{x}{2}\left(\cos \frac{(n+1)x}{2} \cos \frac{nx}{2} - \sin \frac{(n+1)x}{2} \sin \frac{nx}{2}\right) + \left(\sin \frac{x}{2} \cos \frac{(n+1)x}{2} - \cos \frac{x}{2} \sin \frac{(n+1)x}{2}\right) \cos \frac{nx}{2}}{2 \sin^2 \frac{x}{2}} \\ = & \frac{n \sin \frac{x}{2} \cos\left(\frac{2n+1}{2}\right) x - \sin\left(\frac{nx}{2}\right) x \cos \frac{nx}{2}}{2 \sin^2 \frac{x}{2}} \\ = & \frac{n\left(\sin\left(n+1\right) x + \sin\left(-nx\right)\right) - \sin nx}{4 \sin^2 \frac{x}{2}} \\ = & \frac{n \sin\left(n+1\right) x - \left(n+1\right) \sin nx}{4 \sin^2 \frac{x}{2}}. \end{aligned}$$

Hence we have

$$-\sum_{k=0}^{n} k \sin kx = \frac{n \sin\left(n+1\right) x - \left(n+1\right) \sin nx}{4 \sin^2 \frac{x}{2}}$$

and the result follows.