Solution (#93) From #90 we know that $z^3 - mz + n = 0$ has three distinct real roots when $m > 0$ and

$$27n^2 < 4m^3.$$

Recall the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. So if $z = k\cos\theta$ then

$$z^3 - mz + n = k^3\cos^3\theta - mk\cos\theta + n.$$

In order to write this expression as a multiple of $\cos 3\theta$ we need

$$\frac{k^3}{mk} = \frac{4}{3} \implies k = \pm\sqrt{\frac{4m}{3}}.$$

If we choose this positive value of k and set $z = k\cos\theta$ then

$$\begin{aligned} z^3 - mz + n &= 0, \\ \frac{4m}{3}\sqrt{\frac{4m}{3}}\cos^3\theta - m\sqrt{\frac{4m}{3}}\cos\theta + n &= 0, \\ \frac{m}{3}\sqrt{\frac{4m}{3}}\cos 3\theta + n &= 0, \\ \cos 3\theta &= -n\sqrt{\frac{27}{4m^3}}. \end{aligned}$$

As $27n^2 < 4m^3$ then

$$-1 < -n\sqrt{\frac{27}{4m^3}} < 1.$$

If $n > 0$ then 3θ can take values in the intervals

$$\frac{\pi}{2} < 3\theta_1 < \frac{3\pi}{2}, \qquad \frac{5\pi}{2} < 3\theta_2 < \frac{7\pi}{2}, \qquad \frac{9\pi}{2} < 3\theta_3 < \frac{11\pi}{2},$$

so that

$$\frac{\pi}{6} < \theta_1 < \frac{\pi}{2}, \qquad \frac{5\pi}{6} < \theta_2 < \frac{7\pi}{6}, \qquad \frac{3\pi}{2} < \theta_3 < \frac{11\pi}{6},$$

and this leads to three distinct values $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$. A similar argument can be made if $n < 0$.