Solution (#96) (i) If we divide the given equation by 3 it becomes

$$z^4 - \frac{2}{3}z^3 + \frac{5}{3}z^2 + \frac{2}{3}z - \frac{1}{3} = 0.$$

Following #95 we make the substitution $z = Z + \frac{1}{6}$ and the equation becomes

$$\left(Z + \frac{1}{6}\right)^4 - \frac{2}{3}\left(Z + \frac{1}{6}\right)^3 + \frac{5}{3}\left(Z + \frac{1}{6}\right)^2 + \frac{2}{3}\left(Z + \frac{1}{6}\right) - \frac{1}{3} = 0.$$

Expanding this we arrive at

$$Z^4 + \frac{3}{2}Z^2 + \frac{32}{27}Z - \frac{77}{432} = 0. \tag{8.6}$$

(ii) We now look to add $(AZ+B)^2$ to both sides in such a way that the LHS becomes a square. From equation (1.32) we know A^2 is a non-zero root of

$$x^3 + 3x^2 + \frac{80}{27}x - \frac{1024}{729} = 0.$$

Following Algorithm 1.29 we make the substitution $x = X - 1$ and with some simplifying we see the equation becomes

$$X^3 - \frac{1}{27}X - \frac{1726}{729} = 0. \tag{8.7}$$

Again following Algorithm 1.29 we now set

$$D^2 = \frac{m^3}{27} + \frac{n^2}{4}. = \frac{1}{27}\left(\frac{-1}{27}\right)^3 + \frac{1}{4}\left(\frac{1726}{729}\right)^2 = \frac{27584}{19683},$$

so we may take $D = \left(8\sqrt{431}\right) / \left(9\sqrt{3}\right)$ and further set

$$\begin{aligned} t &= -n/2 + D = \frac{863}{729} + \frac{8\sqrt{431}}{81\sqrt{3}} = 2.367626\ldots, \\ u &= n/2 + D = -\frac{863}{729} + \frac{8\sqrt{431}}{81\sqrt{3}} = -7.947523\ldots \times 10^{-7}. \end{aligned}$$

T and U are cube roots of t and u, with the three roots X equalling $T - U$ provided we choose T and U so that TU is real. As we are only looking for a single non-zero root for X we may take

$$T = 1.332818\ldots, \qquad U = -0.009262\ldots, \qquad X = 1.342081\ldots, \qquad x = 0.342081\ldots$$

with x being the required non-zero solution.

(iii) Then

$$A = \sqrt{x} = 0.584877\ldots, \qquad B = -\frac{M}{2A} = -1.013191\ldots, \qquad C = \frac{1}{2}\left(A^2 + L\right) = 0.921040\ldots.$$

(iv) The four roots of equation (1.33) are the roots of the quadratics

$$Z^2 - AZ - B + C = 0 \quad \text{and} \quad Z^2 + AZ + B + C = 0,$$

so that

$$\begin{aligned} Z_1 &= \frac{A + \sqrt{A^2 - 4C + 4B}}{2} = 0.292438\ldots + i1.359673\ldots \\ Z_2 &= \frac{A - \sqrt{A^2 - 4C + 4B}}{2} = 0.292438\ldots - i1.359673\ldots \\ Z_3 &= \frac{-A + \sqrt{A^2 - 4C - 4B}}{2} = 0.129071\ldots \\ Z_4 &= \frac{-A - \sqrt{A^2 - 4C - 4B}}{2} = -0.713949\ldots \end{aligned}$$

Finally as $z_i = Z_i + 1/6$ then the four roots are, to 5 decimal places,

$$z_1 = 0.45911 + 1.35967i, \qquad z_2 = 0.45911 - 1.35967i, \qquad z_3 = 0.29574, \qquad z_4 = -0.54728.$$